POPULAR MATHEMATICS,

BEING

THE FIRST ELEMENTS OF

ARITHMETIC, ALGEBRA, AND GEOMETRY,

IN THEIR RELATIONS AND USES.

BY

ROBERT MUDIE,

AUTHOR OF "THE BRITISH NATURALIST," "THE NATURAL HISTORY OF BIRDS,"
"POPULAR ASTRONOMY," &c.

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PREFACE.

Though the subjects treated of in this volume, have, individually, and more especially in the relations which subsist between them, engaged my attention more frequently, more deeply, and I may add more delightfully, than any other subjects of a scientific nature, which I have made the object of thought; and though very many years have elapsed since I first felt the want and the desire of possessing some such book—and even since I came to the resolution of attempting its production, and had in some sort sketched its plan—yet, I fear, and indeed feel, that the execution of it stands more in need of a preface, or explanation, or apology, than any work which I have hitherto attempted.

I am aware that it is not a book for even the moderately learned in systematic mathematics, far less for those whose talents and acquirements do honour to the science and extend its boundaries. I am somewhat apprehensive, too, that it may not prove to be exactly the book which is to afford to the young and the unlearned the spirit of mathematical science, and the force of mathematical truth—the communicating, or, to speak more correctly, the inspiring, or rather the stimulating of which, is the main object by which I have been guided both in preparing and in executing the work.

My chief ground of apprehension on this latter point is the fact of there being no similar book, by the success of which I could be encouraged, or by the failure of which I could be warned. Thus I have been thrown upon my own conception of what was most likely to be useful, without any direct experience on the part of others, by which I could be assisted or guided. But though I laboured under this, perhaps the greatest disadvantage that an author could have had, I feel that I also enjoyed some peculiar advantages. My notions on the subjects, and on the mode of conveying information respecting them, were originally my own. A disciple of no school, and a follower of no master, I had no mannerism of others to come between me and the truth: and it so happened that almost as fast as I could acquire some knowledge, not of a whole subject, but of the successive parts of it, I had the advantage of trying the effect of what I had acquired, and how I had acquired it, in the instructing of others; and very frequently I found that the indirect lesson which I derived, from the effect produced upon the student, was of even greater advantage to me than what I did for myself, and probably greater than I could have derived even from able instructors. I need not say that these few circumstances are not mentioned in any spirit of boasting or self-gratulation, for truly there is not in them anything of which even the vainest man could boast. Besides, after a man has been hammered, pretty smartly and pretty constantly, upon the world's anvil for half a century, though the metal of his mind may not thereby be changed, yet it is beaten to so much compactness, that there are few pores in it for holding so unsubstantial a thing as vanity. I have mentioned them rather, as unavoidable reasons why this book should be so different from the ordinary books of elementary mathematics; in addition to which I have endeavoured to supply what none of those books singly, or perhaps the whole of them taken together, can supply.

In saying this I do not mean that there are many new truths in the volume, or that there is one known truth stated more clearly than it is to be met with elsewhere. But considering the vast number of such books which it is necessary to study, with profound and patient attention, in order to get possession of all the truths which are necessary for having a tolerable knowledge of even the first elements of mathematics, in the three departments of General Quantity, or ALGEBRA; Numbers, or ARITHMETIC; and Magnitudes, or GEOMETRY; and the portion of life that even this, which after all is only a sort of mechanical labour, must consume, before the student is in a condition for beginning to generalise; it is easy to see that the business or the pleasures of the world, must neccssarily take hold of a vast majority, even of students, before they have arrived at the commencement of this, the truly mental and most useful part of Geometry.

Having felt this very severely in my own case—and there is too much of the bitterness of regret mingled with it to allow me to forget it—I have endeavoured to start with generalisation at the very outset of this volume, and to hold fast by it on every occasion, regardless how much it might break in upon the symmetry of the book, or the smoothness of its execution.

Such being the case, this work is not to be considered as a book of reference, from which particular truths, or formulæ for the solution of particular problems, are to be borrowed, without reasoning, and often I may add without instruction; neither is it a task-book, to be conned by rote in successive fragments, and parroted without knowledge, until active employment of the mind cause them to be forgotten. It is strictly, (that is to say in so far as I can judge of it, destitute as I am of an external standard of judgment,) what its title expresses—"POPULAR MATHEMATICS;" that is to say, a book which is meant to be read through, and which is intended to inspire those who, from too tender age or want of opportunities and means, have not acquired a knowledge of mathematical science, with a general perception of its nature, a feeling of its power as an instrument both of wisdom and of working, and the love of a farther acquaintance with it. Every one who has caught even one little ray of the glorious light of this science, must feel that it is as powerful as it is brilliant;

and that when we come to work our way to the knowledge of things around us, from the sod on which we stand, to the most remote luminary in the heavens upon which two hundred millions of miles will tell as a measurable fraction, will readily admit, that mathematics is not only the line wherewithal to measure, and the balance wherein to weigh, but that it is the wedge to cleave asunder whatever is too gnarly and stubborn, and the lever to heave aside whatever is too weighty for the other apparatus of thinking and executing.

Those who have formed their notions from those nominal mathematicians, who idle with the disjointed bones of the science in the absence of the life, are apt to suppose, and sometimes to say, that mathematical science has a tendency to curb the fancy, and pedantify the mind. Among all the blunders of ignorance there is not one more gross than this; and we might appeal with triumph to mathematicians of every age as leaving recorded in their writings, abundant evidence of the most exalted and expanded imagination, and the most chaste and lively fancy. I shall mention only one or two names; and these of the last and the present generation. Who in his time excelled or even equalled the late John Playfair of Edinburgh, (with whom I have again and again discussed the subject and plan of this work,) in power, in purity, and in beauty of style? And who, in our own times, writes like Whewell or Herschel? Find me the unmathematical man that shall set an idea before the mind, as a mental and tangible solid, with the same power and truth as either of them, and I shall abandon my argument, and join ever after "the herd of gentlemen who write with ease."

So much for the plan and purpose of the work; and the execution can be best seen and judged of in the work itself; therefore I shall only state further that I have been careful to bring forward the three branches in such an order of succession, as that the reader who reads for instruction, (as I sincerely hope many will,) may call them all to his aid whenever he feels it necessary. I have dwelt longest upon those general points which appeared to me to possess in the highest degree the two qualities of furnishing the greatest number of inferential truths and stimulating the reader to seek out those truths; and I have been more anxious to create a love of the science, than to carry the particular departments of it to a great extent. To use a homely simile, if a man gets lamed before he commences a journey, it is far better to cure him and let him start in his own strength, than to carry him half way and leave him in his lameness. But this simile, homely though it is, applies to every branch of education, and to mathematics in an especial manner. To talk about teaching a person a science, is like talking about a lame man's performing a journey when he is carried; but, if we can succeed in awaking the desire and arousing the capacity, the party will learn, not only without our teaching, but in spite of our opposition; and this is the grand object which should be aimed at by every well-wisher to the mental and moral character of the human race.

I cannot say that I shall conclude this preface—for the same train of thought is continued in the introduction—but I shall conclude the present writing by claiming the suffrages of the public in favour of my purpose, how much soever they may blame the execution of it,—only adding, that if the present volume shall meet with a reception at all proportionate to the labour it has cost me, I purpose following it up by another, carrying the three branches of the science as far as they are required by those who are not professional mathematicians.

ROBERT MUDIE.

Grove Cottage, Chelsea,
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POPULAR MATHEMATICS.

SECTION I.

GENERAL REMARKS, AND DEFINITIONS.

The common opinions of mankind upon a subject are frequently at very remarkable variance with the nature of that subject; and this variation is not perhaps more striking in any one case than in that of Mathematics. Those who have never studied any portion of mathematical science, however acute they may naturally be, and however well informed upon other, and in themselves more difficult subjects, generally, if not invariably, turn away from every mathematical expression, as if it were an adder in their path; and even they who, to use the homely but most appropriate expression, have "gone through" that which is called mathematics at the common schools, shake their heads at the subject, with a silent expression of, "These matters are beyond our depth." The conduct of such parties puts one very much in mind of that of the porter in a northern University. This porter was a very "whale" of books, and one of the professors, whose particular attention he claimed, found the supplying of his appetite from the University library no easy task. At length he tried him with "Euclid's Elements of Geometry," to see how far sheer appetite would be able to digest that. The porter came not for an exchange till after two weeks had elapsed; and at last he came, somewhat crest-fallen, saying, "Docter, I hae read a' the wirds, an' leukit at a' the pikters, but it's the maist puzzleanimous beuk I hae seen, an' I dinna onderstand ae wird o't; sae ye'll jeust hae the gudeness to gie me a beuk that has nae As nor Bs in't."

It is probable that some part of this general dread of mathematics may have been occasioned by the reply of Euclid to Ptolemy Philadelphus, the Egyptian monarch. The king wished to know if there was not an easier method of learning geometry than that which was practised in the schools; and the mathematician bluntly, but somewhat ambiguously replied, "There is no *royal* road to geometry." Now, all that was meant by these words was, that geometry must be studied by man as man, and not as monarch; that it must be conquered by the mental exertions of the individual alone, and not by any subjects which he can command, or any armies that he can muster; so that, if we take it in its true meaning, the saying of Euclid is an express declaration, by one of whose judgment no one can doubt, that any man might be a geometer if he would bring his own mind to bear upon the subject; and that in this science, the civil and political distinctions of mankind go for nothing, for it is as open and as plain to the humblest peasant as to the proudest king.

Sixty-three generations of men, at the average allowance of one-third of a century for each, have been born and have died since this reply was given by the Alexandrian geometer; and during this long period, men of all ranks, from the monarch to the peasant, have studied and promoted geometry, and the other branches of mathematical science; but this reply has been brought forward as a sort of bar in the way, not of kingly power, but of intellectual ability, and the consequence has been that, even at this day, and in this country—the foundation of whose greatness has been mathematical science, the great body of the people know less of the principles of mathematics than of those of almost any other subject. And even that love of reading which has of late years been so generally diffused, and which may be made the instrument of so much good, has not embodied anything like a fair proportion of mathematical knowledge, neither have they who have gone about to cherish this love by the multiplication of books of small size and easy price, done anything like justice to the public in this respect. The mathematical tracts which they have produced are few in number, and as to their value, it is to be feared that it is still less.

With the cause of this deficiency we have no immediate concern: but the probability is, that it is found impossible to *compile* mathematical books—to take a little bit of one, and a little bit of another, and tack them together into an amusing miscellany, any page of which may be read with at least some sort of understanding, without reference to the rest. Or it may be that we possess no mathematicians but such as are professionally so; and thus, however able they may be in a professional point of view, they can treat the subject only in a professional manner, and would consider their labours deteriorated and themselves degraded, if they were to abate one iota of the technicality of the schools. Now we are very ready to acknowledge the full value of this technicality, and to admit that every apparent difficulty in mathematics, is essentially a simplification. We do this confidently; because mathematics is, as we shall show by and by, the only portion of science which has hitherto stood, and must for ever stand, impregnable to the mere book-maker; and that no man can put a single pin to this fabric without putting the right one, and putting it in the right place. But still, perfect and beautiful as is this technical structure, and proudly as it towers over the rest of human knowledge, as the noblest conquest and heritage of intellect, and frowning defiance and scorn against every species of imposture, it is too mighty for any but those who are to give themselves wholly up to it. At the same time, as it is the purest exercise of the mind, the real instrument of discernment, that in which the individual must be thrown wholly on his own strength, it is desirable that some portion at least should be accessible by every one who can read, and that this general portion should not be those insulated scraps of the applications which are useful to men in particular professions, but at least as much of the principles, as shall give a mathematical turn to the mind, which is but another name for precision and accuracy of thought.

It may seem paradoxical, but it is nevertheless true, that however ignorant we may be of the forms of mathematics, and how much soever we may regard the technical expressions of the different branches of mathematical science as puzzles or mysteries, we are all mathematicians in reality; and the process by which we arrive at the precise and accurate knowledge of any one subject whatever, is really a mathematical process, whether we know it to be so or not. The only difference, indeed, between one who understands the principles of mathematics, and can apply those principles to the finding of results, and one who must get at the results the best way that he can, without any knowledge of the principles, is, that the first proceeds with ease and certainty, while the other proceeds with great labour, and is doubtful of the result when he has arrived at it. Mathematics, to use a homely comparison, may be compared to tools and the capacity of using them; while the subjects upon which mathematics are exercised are the materials, out of which that which is desired is to be formed by means of the tools; so that a mathematician stands to a man who is no mathematician in the same relation as a clever workman well furnished with tools stands to a man who has no tool and no knowledge of the use of one; and when we look at the accommodations of civilised men, and compare them with those of men at the bottom of the scale of savagism, we are able to judge of the difference between the man who possesses the instrument and knows how to use it and the man who is ignorant of both. The disparity is even greater than this; because mechanical operations, valuable though they be, are only one particular case, whereas mathematics reach every operation of the mind, give clearness to every thought, and regulate with certainty every action.

One other cause of the ignorance in which mankind suffer themselves to remain of mathematics, may possibly be want of knowledge of what the term means; and this is rendered the more probable by the fact that, in the ordinary way of teaching the individual branches of mathematical science, such as arithmetic, or the elements of geometry, the student is sent to the details of the subject at once, and without any preliminary explanation of the use, or even the general nature of what he is called upon to do. The consequence is, that there is no goal before him, nothing to keep alive his hope, or rouse his mental ambition; and so he drudges on like a slave, measuring his labour by the day, and his pleasure by the smallness of the quantity of the day's labour. Upon young minds especially this has a most baneful influence; as it not only destroys the possibility of progress in mathematics, which must either be a labour of the *willing* mind or no labour at all, but becomes a habit, which is transferred to and which destroys every other branch of education, and perverts and poisons every course of future life. How much of this preliminary explanation should be given in any case must depend on the nature of that case itself,—on the age, ability, and previous knowledge of the student; but in every case the danger is that it shall be too little, and not that it shall be too much.

It is perhaps difficult to convey in writing even the simplest outline of what should be done in such cases, because it belongs to the province of colloquial instruction—that in which the instructor can lay hold of present and visible illustrations, and vary, and reiterate again and again, with a tediousness which no ordinary reader would tolerate in print, but which in practice is the only sure way of "trying for the vein" which will make the mine of instruction work easily, certainly, and profitably. The few sentences which follow must therefore be considered, not as furnishing what is to be done, but merely as giving a hint that, in order to insure success, something ought to be done.

MATHEMATICS (Μάθησις) contains in the name itself no bad general definition of the whole science, or rather the *mode*, so to express it. *Thesis* means a position, that which can be either true or false, but it leaves the mode of arriving at the truth or falsehood perfectly general, although it always does involve in it the notion that there is some sort of proof; and as the discovery of a falsehood is a truth, though the falsehood itself is not, every useful thesis may be considered as the statement of a truth; and the truth which amounts to a thesis must not be one which is perfectly apparent to every body without any proof or argument. Thus, if we were to say "It rains," to a person whom we met out of doors during a shower, the saying would be no thesis; neither would it be a thesis to say "The sun shines," to one whom we met in the fields at noon on a cloudless day. In both these cases the party whom we addressed would understand the fact just as well as ourselves, and therefore our observation would be an idle waste of words. But if we were to say "The apparent motion of the sun westward is owing to the real rotation of the earth eastward," it would be a thesis; because it is a truth which cannot be arrived at by simple observation of the sun, but quite the reverse; and therefore, to a person ignorant of the motions of the earth, we would require to bring forward proofs before we could call upon him to believe it. Thus we may consider every truth which requires to be established by reasoning, by evidence, or in any way whatever, as a thesis; and it is not, properly speaking, a thesis until the proof is given, for this simple reason, that no truth can be regarded as such until it is known to be so.

When no proof has been given, but there is still some probability that a position *may* be true, it is a *proposition*, or *hypothesis*, which means something which *precedes*, or is *inferior* to, a thesis; and which requires to be proved before it can be elevated to that character. If the proof shall afterwards be obtained, the hypothesis takes its rank as a thesis, and becomes a portion of knowledge; but if the proof fail, the hypothesis falls to the ground as a vain and unsuccessful attempt.

The methods of proof employed for the establishment of different truths are so exceedingly numerous, that a list of them would be long, not very interesting, and out of place here; but still, in order to see clearly the nature and use of mathematical proofs, or which is the same thing, the mathematical modes of establishing truths and rejecting falsehoods, it is necessary to know something about the general divisions of proof. The simplest view which can be taken of this subject is that which divides the whole into three great classes—observation, testimony, and proof by reasoning. Observation is only another name for that of which we have the evidence of the senses;

but it applies equally to that which we observe as existing, or happening in nature, and to that which is the result of art or experiment. This kind of proof can of course extend no farther than the senses extend; and it must be subject not only to all the natural imperfections of the senses, but to all the effects of carelessness in the use of them. But still, one observation or experiment can be made the means of correcting another; and thus, this species of proof, though it is not so simply and originally, can be corrected and improved, so as to be the standard, or at least the foundation of every other kind of proof. The truths at which we arrive by observation form what we are said to know, "of our own knowledge;" and as this is the standard by which we judge of the second kind of truths, correctness and extent of observation are of the greatest value to us.

Testimony or evidence is our reliance upon the knowledge of others, as told to us in speech or in writing, and it necessarily comprehends a very large portion of all that we know. The kind of truths which can be fully established by evidence are exactly the same as those which we establish by our own observation, with this difference, that evidence gives us a command of time and place which we ourselves cannot personally enjoy. All history, whether of the common events of nature, or of human society, rests upon proof of this kind; and this proof, though it may have a high degree of probability, never partakes of the absolute certainty of what we know of our own knowledge. There are many ways however of correcting testimony, by reasoning on the probability of what is asserted, and also by comparing one testimony with another; but still in this part of our knowledge we stand greatly in need of means of correction; and, both in this and in our personal knowledge, we require to exercise the utmost caution when we attempt to turn that which we already know into an instrument of future knowledge.

Proof by reasoning is of a mixed nature: because we may reason from correct observation, from authentic testimony, or from hypothesis which admits of proof; and we may also reason from inaccurate observation, from false testimony, or from unfounded hypothesis; and we may do so, in the firm belief that all the three rest on the surest foundation. Not only this, but our knowledge of every thing, as derived from personal observation or from testimony, is imperfect, and we are apt to make false comparisons; and thus, with every desire and every endeavour to be right, we are in constant danger of going wrong; and it is to help us in this difficulty that mathematical truth becomes, in a general point of view, and independently of all its practical applications, of so very great value to us.

Mathematical truth means that which is something more than thesis, and which does not depend either upon observation of what exists or upon testimony. It is what we may call absolute or abstract truth; and would remain the same, though all history were forgotten, and every thing of which the senses can take note were destroyed. No doubt it applies to the objects of sense; but still in its principles it is altogether independent of them; and whatever is mathematically true of any object of the senses would be mathematically true to the mind, whether the object of the senses had ever existed or not. And there are many subjects of mathematical science for which there are no counterparts in the world of reality; and yet mathematical speculations grounded on these are as true, and in almost every instance as useful, as if there were an object of the senses answering to each. For example, a mathematical point, the mark of position in space, has no magnitude, that is, occupies no space, and therefore it can have no real existence;

and yet it is one of the foundations of geometrical reasoning, and one without which we should be unable to perform even the simplest operation in measuring. So also a mathematical line has neither breadth nor thickness, and therefore has no more real existence than a point has, but merely marks direction and distance in space in the same way as a point marks position. Farther we may say, that there is no such thing in nature as a perfect circle or a perfect square, neither can we make one by art; and yet upon these figures there is founded a very great deal of useful truth, both in mathematics and in the application of mathematics to other matters. We need not add that there neither is nor can be in nature any thing or quantity less than nothing; and yet, as we shall see by and by, the consideration of quantities in this light is of as much value in mathematics, and leads as certainly to the truth, as the consideration of quantity as being greater than nothing.

Mathematics admits of so many and so varied applications that it is not easy to give a technical definition at once fully descriptive of the subject and useful to the reader; and it is upon the whole to be doubted whether short technical definitions of very general subjects are not as often productive of harm as of advantage. When we have said that mathematical truths are perfectly independent of place, time, and material existence, that they stand in no need of observation, experiment, testimony, or argument, but that they have an inherent power of producing conviction, which nobody that understands, or that ever shall understand them, can by possibility resist, and that thus they can occasion no doubting, and lead to no disputes, we have perhaps said as much as can be said in the way of general definition. With the individual parts it is however very different, for the definitions of them can be made as precise as that of the general science is vague.

But it happens in this case, as in many others, that illustration supplies the place of definition, and this is always an advantage, because, while definition gives us words only, and we are in some danger of resting satisfied with them, illustration leads us to the reality.

Now, for the purpose of illustration here, let any one think over the number of subjects of which he has, or, which answers our purpose quite as well, fancies he has, a competent and even a correct knowledge; when he has done so let him consider of what number of these subjects he has acquired all the knowledge that he possesses by his own unassisted observation and experience, and he will, if he has not made the estimate before, be perfectly astonished at the small number of the latter. Next let him tax his memory to bring to remembrance all that he has seen; and, however short his life may have been and however observant his habit, he will be struck with the singular fact, that he has not one recorded observation which he can fairly set down as a new and separate truth for every day, or even for every year of his life; and that, of the few that he does remember, there is not one in a hundred which he understands half so well as he does many of those other matters of which he has had no personal observation whatever. Farther than this, when we think over these subjects, and attempt to turn them into the means of knowledge of whatever kind, we find that those of them of which we have read or heard come back to the memory as vividly, and if they be scenes, have all their parts as well brought out, and all their colours as warm and rich as those which we have personally visited; and if the scenes have been rendered and the characters delineated with sufficient force and truth, we are absolutely more familiar with them, even if they never had any real existence, than we are with places which we have actually visited, and persons whom we have actually seen. If we change our place of residence, and have no personal cause of remembering the place or the people, they fade away from our memory in brief space; but nobody ever forgets Robinson Crusoe and his man Friday and the island; and nobody that has read Scott's novels ever forgets the Baron of Braidwardine, or Meg Merrilies, or Balfour of Burley, or Jeanie Deans pleading for her sister before queen Caroline.

Now why, we may ask, are these romances so much more deeply impressed upon the mind, and so much more ready and vivid to the recollection, than hundreds of realities that have passed under our notice every day? Many will answer, "it is because they are so natural, so true to nature." We rejoin, that cannot be, for they cannot surely be more natural than nature itself, and yet they are remembered, while real, positive, practical nature, scenes in which we ourselves played a part, and had our feelings interested at the time, are forgotten. And if they are natural, what nature is it that they represent, and where have we seen the original of one of the characters above enumerated? The charm consists in their not being real persons, but merely the mental embodiment of persons; and because they are so, all the relations between action and action are perfect, and the whole character is before us, which it cannot, in the nature of things, be in the case of real beings, as they have the power of concealing part, and actually exercise that power. If these delineations were copied tamely after individuals, they would be admired only by those who knew the originals, and by them only in proportion as they admired them. It is because they are general that their force is felt by everybody; and it is for the very same reason that so much of what we read and hear makes an impression upon us in comparison with what we see.

When we read or hear, the subjects of which we get the information are not before us, we merely have the abstract mental conceptions of them expressed by certain conventional symbols, the letters of the book, or the sounds of the words which are spoken; and it is because the mental conception comes home at once to our own minds, without any of that laborious examination in detail which objects of the senses require, that the knowledge is both so powerful and so permanent. In the real object we can see but one side at a time; and the story of that which we observe does not extend either way beyond the time of our actual observation, without drawing our attention from the reality. But the mental embodyment is transparent: we can see all sides of it at once; we can view it in all its succession of states; and we can bring it forward and study it whenever we please.

Every one must see the advantage of thus being able to bring the mind to bear upon the whole of a subject, in all its parts, in all their connexions and relations, and in succession of time; because we are thereby enabled to "see the end from the beginning." In the forming of any plan we can not only see whether the ultimate object is attainable, but we can see the shortest way to it, press into our service all that can promote its success, and remove all that would occasion hindrance or failure; whereas those who can take no such mental view, but must at every step "wait till they see," are constantly bungling and blundering, and really have more trouble in correcting their own errors than in all the rest of the business of life. We do not mean to say that every one who has this capacity of forming skilful plans, either does form them or carry them into execution; but it is abundantly clear that the capacity must exist before the plans can be formed.

It is not to the projecting, or planning, of any one thing that this capacity applies; for it applies to every thing, from the greatest action in which men can be engaged to the simplest project of the humblest individual; and it applies as well to the conduct of the execution as to the formation of the plan; so that it is in truth the grand rule of life. If it were possible to impart this capacity to every individual, or rather to train every individual to this habit, (for after all it is merely a habit, and a habit which calls for no extraordinary power of mind, but would adapt itself to the mind of every individual,) it would cost less labour in the learning and the observance than people must necessarily undergo in consequence of the want of it, and not only so, but the life of man would in point of efficiency be greatly lengthened. As things are managed at present, a large portion of the time of most people is occupied in returning from the wanderings, and rectifying the blunders of the rest of it; and it is perfectly evident that, if they could be spared this useless labour, they would have all the time it occupies as leisure, during which they might increase their gains, or improve their minds, or enjoy themselves, just as suited their fancies; and thus, while there would be an end of much needless labour and real suffering, there would be a corresponding increase of efficient labour and real enjoyment; for it must be remembered that the value of labour is not measured by time, but by productive power; and that the enjoyment of labour is never so sweet and so satisfactory as when we feel that we have earned it by doing our duty to ourselves and our country in the most perfect and effective manner.

This mental power, in the extended sense in which we have viewed it, cannot be said to be mathematics or mathematical; because it applies to all subjects, while the subjects of mathematics are limited. But the difference is one of subject rather than of principle, and the conduct of the mind in the cases alluded to is in strict accordance with its conduct in mathematical investigations. Indeed, there is so much of similarity, that if we take out those portions of any particular case which are contingent, all that remains is strictly mathematical. The contingent parts are the data obtained from without—the results of experience and of testimony; and though these, taken in the general sense, are not mathematical, yet that keen scrutiny into the truth or the falsehood of everything which a mathematical habit produces, is of great use in estimating the worth both of observation and of that which rests upon evidence. Thus we are to consider the strictly mathematical habit in the mind, which is the most valuable part of the science, to consist in arranging according to their proper relations all deductions which the mind itself is capable of drawing from self-evident principles, and in discovering truth and detecting error in every possible combination which can thus be formed. In the doing of this, there can be no half measures; for to what extent soever we may arrive in the combination, we must be able at every instant to look back to the very outset and make sure that every single step, whatever may be the number, has been taken rightly. Thus, for instance, if the subject of immediate inquiry is the distance of the sun, we must be able to see our way backward from this great distance to the measuring of a straight board with a foot rule, and how we have been able to find our way from so short a measure to one which is so vast. In like manner, if the question be to what height the attraction of the moon shall raise the tide of the ocean, we must be able to connect this with the weighing of a pound or an ounce by the means of a common balance. Generally, whenever we are to apply our mathematics to the ascertaining of anything which we wish to know, however great or however complicated, we must see our way not only from some operation which we ourselves could actually perform, but from some principle so simple and so clear that nobody who understood the words could refuse to give their assent to the truth.

These are great advantages—advantages which, in a mental point of view, we seek for in vain in any other department of human knowledge; and it will readily be admitted that if we are trained and habituated to this extent and transparency of mental vision upon a number of subjects, we will endeavour as a matter of course to exercise the same, as far as it may be practical, upon every subject. Then in addition to this there are the practical applications, which include all calculating, and weighing and measuring, and comparing, and estimating, and determining value of every kind. In a word, if we take mathematics from all the practical knowledge which we find of so much use to us in the occupation and business of life, we should leave nothing behind but uncertain guesses and conjectures, and could not by possibility be either a successful or civilised people. Thus we cannot, and whether we know it or not we do not, manage matters without the virtual aid of mathematics; and why should we not get the real aid? There are no doubt a great many technicalities, and mathematical writing has the appearance of being in a strange tongue. But this is a mistake; the difficulty is not so formidable as it seems, and the language is not only our language, but the language of all nations who will give themselves the trouble of learning that which every child learns first, namely, an alphabet.

SECTION II.

SUBJECTS, OBJECTS, AND PRINCIPAL DIVISIONS OF MATHEMATICS.

Quantity is the subject of all mathematical investigations and proceedings, whether *theoretical* or *practical*, that is, whether relating to the discovery of general principles and relations, or to the application of these to particular cases. Therefore, before we can enter upon the science with any chance of success, it is necessary that we should clearly and perfectly understand what is meant by quantity.

Quantity, from the Latin *quantus*, literally means "as *much* as there is;" and it is easy to see that the words "many, large, great, long, quick," and an endless variety of others, may be used instead of the word "much," or that the word "much" may be retained, and the other word added to it, being at the same time changed to a noun. Thus, "as much of largeness as there is," and so in all cases. But we cannot thus turn the word "much" into a noun, and use any of the other words that satisfy the meaning of the sentence when alone as an adjective before it. Thus we can see that the word "much" is a more general one than any of the others, and can be applied to every kind of quantity, while the rest apply to particular kinds or particular modifications.

We could with equal propriety use the word "little," which refers to quantity in the same general sense as "much" does, and the difference between them is a matter of relation and not of reality. This will readily be perceived when we consider that the very same quantity can and would be considered as much by one party and little by another. Thus five pounds in money would be much to a poor labourer for a week or even a month of hard toil; but a lawyer in first-rate practice would consider it little for speaking some hundred words which cost him no labour at all. A wagon and horses would not only be much for a man to carry, but too much for being carried by a score of the strongest men in any parish. But all the wagons and horses, and all other things, moveable and immoveable, on the surface of the earth, are so very little for the earth to carry, that they do not in the least hinder its motion round the sun, which is at the rate of more than seventy thousand miles in the hour.

As the words much and little are thus equally expressive of quantity, the simplest and most general definition of quantity which we can obtain is, that which we can call either much or little.

It will follow from this definition that there is almost an endless variety of quantities, not only of individual quantities but of kinds of quantities; and these quantities consist, not only of things, but of the relations of things, and of all sorts of changes and successions, whenever we can call them either much or little. Thus time is a quantity, mere distance from place to place is a quantity, motion is a quantity, and even the change of motion is a quantity: for, in respect of time we can say, "It is much longer;" of distance, "It is much greater;" of motion, "It is much quicker;" and of change of motion, "It is much quicker now than it was before." We can also apply the word "little," or some word having a similar meaning, in each of these cases, and therefore they answer the whole definition of the word quantity.

In order to be able to use quantities in practice, it is necessary that we should have the means of answering the question, "How much?" or "How little?" with regard to them; and thus the next consideration is, how this is to be done. In the simplest view of the matter, we may suppose the quantity respecting which we wish to answer the question to be known as a whole; as, for instance, how much money is in the purse, how much measure is in the table, and so of other cases. Now it will be immediately felt that in order to answer these questions, simple as they are, there is an element wanting; for when we put the question, "How much money is in the purse?" another question immediately rises to the mind, and demands an answer before the first one: "What kind of money?—sovereigns? shillings? or what?" If we took any quantity whatever, a similar question would arise; so that in all cases where we asked, "how much?" we would be met by the question, "of what?"

The answer to this question must be made in something that we know already; for if not, the very same question would arise a second time. Let us take an instance: "How much money is in the purse?" "Of what money?" "British money." And then comes the question, "What *kind* of British money?" and if the answer be, "sovereigns," or "shillings," or anything else that we name, and know as a kind or denomination of British money, we are in a condition for getting an answer to the first question, but not till then.

The shilling or sovereign, or whatever else the denomination may be, is the *standard* or *measure* which we are to apply to the money in the purse; and in the case of every quantity, we must have a standard or measure before we can find how much the quantity is, and this measure must be known to us, and must be of exactly the same kind as the quantity. This, though a simple consideration, is an important one, and it may not be amiss to see what would be the effect of referring to a standard not of the same kind with the quantity we intended to measure. Perhaps none is better than the traveller's ironical question to the Irish ditcher, and the ditcher's reply. The traveller was going toward the town of Mullingar, and asked a ditcher by the road-side, "Pray, my good fellow, how far is it from Mullingar to Martinmas?" "Plase you, yer honour, and it's just as far as from Christmas to the ace of spades."

But we have not only to determine how large single quantities are in terms of some known measure, for we often have occasion to compare the whole of one quantity with the whole of another, without any reference to the particular measuring of either of them, and therefore it becomes necessary to have some general means of determining when quantities are of the same kind with each other and when they are not. In the case of quantities which really exist and are palpable to the senses, we are never at much loss to find out, at least in a general way, which are of the same kind with each other and which not; but in our mathematical inquiries, we make use only of the relations of quantities and not of the actual quantities themselves, and therefore it becomes necessary that we should have a standard whereby to determine generally when they are of the same kind and when they are not. Now the simplest test of sameness that we can have is that of being able to say that the quantities are either equal, or that the one of them is greater than the other; and simple as this seems, it is all which we require, only we must be careful to view each of them in its whole character, and not to estimate both in any one quality which is common to the two. Thus if the comparison were one hour of time and four miles of a road, and it were asked whether these were of equal length, or which were the longer, no answer could be given, and the quantities are clearly not of the same kind. If, however, we referred the hour and the four miles to something travelling along the road, they might be equal, or either of them might be the greater. For instance, to a man walking four miles an hour, the hour of time and the four miles of road would be of exactly the same length; to a coach running twelve miles an hour, the hour would be three times as long as the four miles of road; and to a pig getting on at the rate of a mile an hour, the four miles of road would be four times as long as the hour. Thus when we speak of quantities, as being of the same kind or of different kinds, it must always be understood that we speak of them as independent quantities and not as relations of other quantities, even though they can exist only in the latter sense. Thus the strength of a man, the strength of a horse, the motion of the wind, the weight of falling water, and the elastic force of steam, are in one sense all quantities of the same kind, and the effects of them, and consequently they themselves, admit of being measured by the very same standard, though the substantive existences are all different, and no two of them admit of any comparison.

All quantities, which can exist, or of which we can form any notion as being either much or little, whether we can express them exactly in terms of any standard or not, can become subjects of mathematics, and so can all those relations of quantities to each other which we can in any way understand or express; and when we speak precisely of a quantity, or name that which we call the *value* of it, we always name a relation—the relation which it bears to the known standard in which we estimate that kind of quantity. In quantities of daily occurrence, we generally have a considerable number of standards or denominations, as we term them; as in money we have pounds, shillings, pence, and farthings, and any one quantity of money we can express in any one of those denominations with equal accuracy, though for convenience we express large quantities in the larger ones, and small quantities in the smaller, and the relation of any known or measured quantity to the standard in which it is measured is expressed by a *number*. Thus five pounds expresses the relation of a known quantity of money to a pound, considered as the standard; and it is of no consequence as to mathematical value, whether this five pounds be five pieces of gold coin, each equal to a pound, or anything else which would at the public market readily and always exchange for those five pieces.

In the case of two such standards, as, for instance, one pound and one shilling, it is evident that we can express the value in terms of either of them, provided we know the relation between them; and from the common way of measuring by means of a standard, with which everybody is acquainted, it will be perceived that the relation of any quantity to any other of the same kind is the number of times that the first is contained in the second. Thus the relation of a shilling to a pound is one to twenty, and the relation of a pound to a shilling is twenty to one; and it must be understood in all cases that the number which results from this comparison is the measure or value of the second, in terms of the first or standard, considered as one whole. So also, upon the same principle, it is evident that if any two quantities of the same kind are in this way compared with a standard, which is the same in the case of both, the results of the comparisons with this standard would accurately express the relation of the two quantities to each other. This is called the [redacted] *ratio* of the two quantities, and though simple when viewed in this light, it is, in a practical point of view, one of the most important principles in the whole range of mathematical science. We must readily admit this, when we consider that we can obtain no knowledge of the value of any thing but by referring it to some standard which we already know; and not only this, but that we can get no knowledge of anything whatever but by comparing it with what we already know. This comparison, this finding of the *ratio*, or *relation*,

~~or proportion~~, between one thing and another, is therefore the most valuable of all operations; the only means which we have of understanding it fully is the study of mathematics, and did that study lead to nothing else, it would be worthy of all the attention we can bestow upon it.

But we have not only to examine and understand quantities and the relations of quantities which can or which do exist; we have to consider those that have no existence, and even those of which the existence is utterly impossible. Every plan or scheme which we form is a quantity which does not exist until we have put it in execution; and, even with every desire and every effort on our part to carry those plans into execution, they very frequently fail because they involve impossible quantities of which we were not aware when we formed them. In the common business of life, where we have not all the elements under our controul—within ourselves as it were, but must be controlled by other people and by the general events of the world, which never give us full warning of their coming, we cannot in the nature of things avoid all these impossible elements; but still it is of the utmost advantage to be "in the way" of doing it, and there is nothing which puts us so much in this way as a mathematical habit—that of estimating the value of every circumstance and every probability in terms of some known standard.

Even when we ourselves have or should have perfect controul over all the elements which enter into our scheme, there is often some lurking impossible quantity which insinuates itself into the chain, and mars the purpose of the whole; and we may be sure that when any scheme fails, without negligence on our part or prevention from any external cause, there has been an impossible element in that scheme at its formation. Of the vast number of inventions and projects which are every day brought before the public, not as mere bubbles or impostures, but with perfect honesty and zeal on the part of the projectors, we speak with most charitable liberality when we say that not one in the hundred proves to be of any use, and nine out of every ten are altogether impracticable. The reason clearly is, that neither the projectors, nor those by whom they are encouraged, are able to see the impossible elements which their schemes involve; that they look at the possible and promising ones only; and thus a large quantity of well-meant labour and ingenuity is constantly wasted.

It may be worth while to mention one case by way of illustration; but before we do this, we may mention that every scheme, process in reasoning, and other plan or project, whatever it may be, is like a machine, no stronger than its weakest part, or like a chain, of which if one link is broken the whole is broken. This consideration is as universal as it is important, and it is the want of attention to this which causes so many sad failures after long and arduous labour with every prospect of success. If the study of mathematics (we speak of the mode of mathematical reasoning, and not of any one branch or application of the science) had no other value than that of enabling us to detect the one cause of failure amid the thousand prospects and promises of success, the time and labour bestowed upon the study would be amply repaid; but this, though a great and perhaps the greatest advantage, is an indirect one, and accompanies the others without our pursuing it as one of our specific purposes, at least if we go to the general principles of the science, and do not confine ourselves to the mere details and mechanical operations.

Now for our case in illustration:—Perhaps we cannot select a better one than that of the "Perpetual Motion," that is, a self-moving machine which shall not involve any cause of stoppage save the wearing out of the materials of which it is composed. We believe that the fonder votaries of this visionary project do not take even the wearing out of the materials into the account; but it is necessary to do this; and even this necessity, when analysed, involves the necessity of the machine stopping before the parts are worn out.

It may be useful to those who are not acquainted with the method of analysis, or of separating the parts of compound subjects and estimating their values singly, and not taking them simply in the mass, when the favourable ones are sure to hide the unfavourable, to point out by how little either of thought or of trouble we arrive at the truth of this case. Every thing on or near the surface of the earth gravitates towards the earth's centre, and this gravitation, which depends on the mass or quantity of matter of the earth, is a power acting constantly and uniformly, so that the tendency of it is to bring every piece and combination of pieces of matter to a state of rest. Thus if an animal walks or a wheel rolls along, this gravitation is continually attempting to stop it, in the proportion of the weight of the animal or the wheel. It is this gravitation which causes animals to make hoof-prints and wheels to make ruts in soft ground; and though we make the surfaces ever so hard or so smooth, though we could get rid of the mere friction or rubbing as not arising from weight, yet the weight is not thereby lessened, that is, the gravitation toward the earth is not one jot the less. Thus in the case of a rail-road, the same power can pull far more upon a level than where the surface is rough or soft, or both; and if there is even a very small declivity, the weight alone will bring down the load without any pull, and bring it down the faster the smoother that both the rails and the wheels are. But when we come to an ascent, even a very trifling one, the disadvantage is just as great, and one horse would pull more up the natural slope of a hill, than a steam engine of a thousand horse' power could do upon wheels and rails up the same.

The power which tends to stop the motion of all machines upon the earth's surface is, then, a power which acts constantly and uniformly, never pausing an instant, nor abating a jot; and therefore, in order to get the better of this gravitation, we must have a counteracting power as continually new as itself; and we are not acquainted with any such power, or any kind of matter in which such a power could reside. It is not difficult to calculate (upon mathematical principles), that if we could give any piece of matter a motion round the earth at the rate of about five miles in a second, or one thousand eight hundred miles in an hour, and *keep up* the motion at this rate, we should overcome the gravitation of that piece of matter. This is what may be regarded as the possible case of the perpetual motion; in this case, the piece of matter must move round the earth, and in no other direction, and it must move unconnected with anything else; and, taking all these circumstances into the account, it will be admitted that the accomplishment is hopeless, and would be useless if it were not.

In the case of a fixed machine,—and the more complicated that the machine is, it is the less likely to succeed,—the impossible element, in the most simple view we can take of it, is this:—to find a piece of matter which, of itself, shall be alternately greater and less than itself, and which shall also remain equal to itself all the time; and if this is not an impossibility, it is not easy to see where impossibility is to be found.

The knowledge of impossible or absurd quantities, and the method of readily discovering them, are often of great use to us, not only in preventing us from wasting our time in attempting to do that which cannot in the nature of things be done, but in enabling us to prove or demonstrate truth in cases where that cannot be done directly; for it is easy to see, that if an impossibility or an absurdity would be the necessary consequence of anything else than one particular state of things, then that particular state is the true one. This method of proof is, of course, not so simple as the direct method, but it is often not less convincing; and we shall see afterwards that, in many cases, it is the only species of proof which we can obtain.

The use of Mathematics, as a general exercise for the mind, and a general guide to the art of thinking correctly, may be in part seen from what has been stated in this section; and the more direct and immediate uses of the different parts can be better explained when we notice those parts themselves; therefore we shall close this section with the names and very short definitions of the principal branches into which mathematical science is divided. Of these, in the very simplest view of the matter, there are three:—

First, ALGEBRA, or the science of quantity in its most general sense, applying equally to every quantity, whatever may be its nature, and whether possible or impossible; and also to all relations of one quantity to another; and being, on this account, the proper foundation of the whole.

Secondly, GEOMETRY, or the science of extended quantity or magnitude; that is, quantity considered as existing in and occupying space. Geometry is thus a particular branch of that general science which Algebra comprises; and though, so far as Geometry extends, both it and Algebra may be applied to the very same quantities, yet geometrical quantities are always such, that we can imagine them to exist and be visible, which is not the case with all quantities to which Algebra applies. It very often happens, however, that the very same mode of reasoning applies to quantities which have a geometrical form and existence, and to those which have not. Thus, for instance, the globe of the earth, considered as a piece of matter of a certain form and magnitude, is not only a geometrical quantity, but the very name, Geometry, means "measuring the earth" (it originally meant what we now call land-measuring); but the attraction of gravitation, by means of which bodies fall to the earth, and are retained on its surface, is not in itself a geometrical quantity, because we cannot say that it has either size or shape, and yet the law according to which it acts is a geometrical law. Thus all geometrical quantities must be such as that we can imagine them to exist in space; but it is not necessary that they should actually fill any portion of that space. Thus, the surface of the table is a geometrical quantity, and so is the length or the breadth of the table; and these quantities are so related, that we can find the extent of the surface if we know the length and the breadth. But none of these quantities occupies any space, for the surface of the table merely separates the table from the air over it, and the length and breadth are mere expressions for how far it extends in two directions across each other.

Thirdly, ARITHMETIC, or the science of quantities expressed in numbers, either exactly or as nearly so as may be possible. This is the practical application of both Algebra and Geometry; and while those sciences express quantities in a general manner, and in such a way as that any conclusion at which we arrive concerning them, is equally applicable to all quantities of the same kind, Arithmetic takes with it the particular values of quantities; and thus arithmetical conclusions have not that general character which belongs to Algebra and Geometry.

Each of those great branches of mathematical science admits of many subdivisions, according to the nature of the quantities, and the relations in which they are viewed; and it may be said, generally, that the grand object of Algebra and of Geometry, besides their great use in teaching the art of accurate thinking, is the preparation of all subjects of which the values can be expressed in numbers, in such a manner as that we can apply Arithmetic to them, and thus ascertain their real values in terms of that known standard by which we are accustomed to measure the kind of quantities to which they belong.

In a civilised country, there is nobody so humble or so illiterate as not to have occasion for a little arithmetic; that is, to be able to express the values of a few quantities in terms of some standard, and therefore a little of the practice of Arithmetic forms a necessary part of every body's education, whether it is acquired at school, or picked up by ourselves in the same way as we learn to speak, and whether it is or is not accompanied by the capacity of reading and writing. Such arithmeticians do not, however, understand any of the principles of that science of which they can thus make a little use; neither are they aware of the advantages which they derive from the science, even in their humble way. It is a fact, however, that the inhabitants of countries in which there never has been any science, or any scientific men, find counting, even to a very limited amount, an operation altogether beyond their power. It is generally said, that many tribes of the North American Indians, when they were first known to Europeans, were quite incapable of counting beyond the number three; and yet it is admitted that these tribes were exceedingly shrewd people, and much more dexterous in the use of their senses than the peasantry of civilised countries. Indeed, even if we take those beginnings which are obtained in our own schools, and in consequence of which the possessor is considered qualified for being a counting-house calculator, we should find it to be exceedingly difficult to arrive, by means of them, at the establishment of any one arithmetical truth, to say nothing of truths of a more general nature; and therefore, in order to understand the principles, we must make another and a more general beginning.

But, in order to do this properly, it is necessary that we should understand the simpler operations of Arithmetic—the way of expressing quantities arithmetically, and of performing on them those few general changes of which Arithmetic admits. This is necessary, for the very same reason that it is necessary to learn the alphabet, the spelling, and the words of a language, before we begin to study the grammar of that language, so as to understand its structure, its power, its beauty, and its deficiencies, and make ourselves master of its spirit and its extent, so as to express what we wish to say or write in the clearest, most forcible, and most impressive manner; and perhaps it is as desirable that we should not attempt to mix up any of the principles with the learning of this first and simplest alphabet of Mathematics, as it is to avoid confounding the infant which is drudging at its Christ-cross row, with lectures about adverbs and pronouns. We shall assume that the least informed reader whose attention is drawn to this volume, is in possession of this arithmetical alphabet, and of a good deal more, and consequently we shall pass very lightly over this part of the subject.

SECTION III.

ARITHMETICAL NOTATION, AND SCALE AND DISTINCTIONS OF NUMBERS.

Little as we are accustomed to think of our common arithmetical notation, and lightly as we esteem the value of that classification of numbers which it represents, it is really, (second only to the alphabet of language, and second to that only because it is more confined in its application,) among the most wonderful contrivances of human ingenuity. Let us take an instance, and consider in what situation we should have been placed, if we had been without our arithmetical scale of arrangement, and our corresponding method of notation, or expressing numbers by a limited number of characters applicable to that purpose, and that purpose only. From what has been said of the state of the American Indians, it is not at all probable that we could have had any means of arriving, not at the knowledge only, but even at the name of any such number as we are to instance; but, for the sake of the argument, let us suppose the thing possible. Well, the average distance of the sun from the earth, expressed in words according to our scale of numbers, is, ninety-five millions of miles; and the same in the notation of Arithmetic is, 95,000,000. Both of these expressions are very short; but if we had had no system, and so had been obliged to express this distance by a repetition of a separate name for every individual number, from 1 to 95,000,000, these names alone would have filled nearly four hundred volumes of about the same size and style of printing as the present one; and therefore, to have made any use of the number, or even to have formed any guess respecting its nature or amount, would have been wholly out of the question.

It is worthy of remark, as a proof of the value of Mathematics, even in the very alphabet of Arithmetic, that we are enabled to get the better of the difficulty by means of what we may strictly call a geometrical principle. Thus, in the expression

$$1\,1\,1\,1\,1\,1\,1\,1\,1,$$

each of the characters has a different value; this value is smallest in the character nearest our right hand, and it increases at a regular series of *ten times* in all the others. The second from the right is ten times the first, the third ten times the second, the fourth ten times the third, and so of the others, whatever may be their number; for as there is nothing to limit it except space to write the characters on, we may make it as extended as we please. This character, while it preserves the same form, and always means *one* of something, may thus have an endless variety of meanings, according to the characters that are on the right of it; and it is evident that it is the number of those characters only, and not their particular values considered in themselves, upon which this second element of the value of the character depends. Thus, if we take a character which in itself stands for no number, and thus marks place in the expression, but not value, we shall be enabled to express any one of the above characters singly, with exactly the same value as it has in the combination. What character we use for this purpose is of no consequence, provided it be one to which we never attach value as a number; and thus 0 has been used, probably because it is a character which can be very easily written. By means of 0, we can express the above nine repetitions of the character 1, by the following nine expressions, all of which taken together would have exactly the same value as the expression from which they were all derived. Thus:—

100000000, 10000000, 1000000, 100000, 10000, 1000, 100, 10, 1.

This is what we may term the system or series of the scale of numbers, and the root, or ratio; that is, the relation of any one term, or place, to the one next it, is *ten*. But we must understand that it is not the number ten counted in individual ones, as we count when we wish to know "how many;" it is *ten times* greater, if the place is next on the left hand, and *ten times* less if it is next on the right.

But as all numbers, considered merely as such, and without any reference to real quantities or things which actually exist, must be considered as quantities of the same kind; and as such we require to have one common measure, or standard, to apply to them all, and in terms of which we may understand them all to be expressed. The number 1 is at once the simplest and the best which we can use for this purpose; and we apply other names expressive of the other places, or terms, of the scale, as measured by the number 1. Thus, 10 is ten, 100 is one hundred, 1000 is one thousand, 10000 is ten thousand, 100000 one hundred thousand, 1000000 one million, 10000000 ten millions, 100000000 one hundred millions, and so on till we come to 1000000000000, which is one billion, it being understood that the words "of 1's," or "of times 1," may be added to the name of each number.

From this structure of the scale, it is evident that, besides 0, which marks place only, we require but nine characters, and nine original words, in order to be able to note, or mark down, and to name, all numbers whatever. These characters are, in their order from 1 to 9 inclusive,

1, 2, 3, 4, 5, 6, 7, 8, 9;

and every character in a number expressed by more than one character, is understood to express as many times the value of 1 in the place which it occupies, as it stands for when original and alone. Thus, in the expression 365, 3 is three times 100, 6 is six times 10, and 5 is five times 1; and so in all other cases. The number 1 may be called a *unit* or *unity*, because it represents that which is entire, simple, and not, in our common notion of it, in any way made up of parts; and from this, the right hand figure of every number may be called the *unit's place*, and is the beginning, or rather, as we shall see afterwards, the middle of the arithmetical scale; and, for a reason afterwards to be mentioned, it were perhaps as well that we should affix a point (.) on the right of every number ending at this place, independently altogether of the common use of the same mark for dividing language into sentences.

From what has been said, it necessarily follows that it is only those figures or characters, in numbers, which are equally distant from the unit's place, which can be considered as quantities arithmetically of the same kind, or having 1 in the one of them exactly equal to 1 in the other. It is necessary to attend carefully to this in all arithmetical operations, because, when we come to Arithmetic, we are always understood to deal with the exact values of things; and we cannot strictly call one thing and another by the common name two, unless they are precisely of the same kind. Thus, though a bird and a quadruped are two beings, and even two animated beings, they are not two of any one kind of beings.

As each place or term in the scale of numbers is obtained by taking ten times the place or term on the right of it, or, which is the same thing, multiplying the term next on the right by ten; so conversely we may obtain the next term to the right by taking a tenth part of any term, that is, by dividing it by ten. When we take this view of the matter, it immediately becomes apparent that there is no necessity for stopping our scale of numbers at the unit's place; but that we may extend it downwards without limit, as well as upwards, and still preserve the same ratio, or relation, of ten times as taken upward, and one-tenth as taken downward, between term and term; and this is what is called the complete DECIMAL SCALE of Numbers, or of Arithmetic, from the Latin, *decem*, ten. An intimate acquaintance with this scale is of the utmost consequence to every one who wishes to use Arithmetic readily, easily, and correctly, even in the most common business of life; we shall, therefore, make one or two further remarks on it.

On looking back to the analysis of the number 111111111, into the nine numbers of which it is composed, it will be perceived that every tenfold increase, or multiplication by ten, is expressed by the adding of one 0 to the next lower term; and if we begin with the highest, we perceive that every term is reduced to one-tenth of what it was, by taking away or subtracting one 0 from it. We may mention, in the meantime (for we shall explain afterwards), that when the operation of adding is expressed, but not performed, this is done by prefixing to whatever is to be added the sign +, which is called *plus*, and may be read "more;" and that when subtraction is expressed, but not performed, it is done by prefixing to whatever is to be subtracted, the sign —, which is called *minus*, and may be read "less." Thus, $5+2$, is 5 and 2 more, or 7; and $5-2$, is 5 and 2 less, or 3.

When we represent the terms of the scale of numbers, and use the number 10 for each of them, we bring it into shorter compass, with still the same meaning, if we use the common figures or numerals instead of the numbers of 0's; but we must write them in a different form, so that they may not be confounded with the common figures of Arithmetic. They are usually written, in much smaller characters, over the right hand of the other figures; and, for a reason which will presently appear, they are called *indices*, *exponents*, or *exponential numbers*. Thus, instead of 1 eight 0's, we may write 10^8; instead of 1 seven 0's, we may write 10^7, and so of all the others, as follows:—

$$10^8, 10^7, 10^6, 10^5, 10^4, 10^3, 10^2, 10^1, 10^0.$$

It must be carefully borne in mind, that these exponential characters which are placed over the right of the others, do not mean numbers, either as added to or as multiplying the others, or as in any way connected with them. They mean numbers of times multiplied by 10; the right hand one, 10^0, not multiplied by 10 at all; the second, 10^1, once multiplied by 10; the third, 10^2, twice multiplied by 10; and so of the others. If we take them from right to left, we find that the exponents increase by the constant addition of 1; and if we take them from left to right, we find they diminish by the constant subtraction of 1; the addition of 1 in the one case, being equivalent to a multiplication by 10; and the subtraction of 1, in the other case, equal to a division by 10. We may, therefore, continue our subtraction of 1, and our series, as far as we please, only after 1^0 we can only *indicate* the division by the sign —. Thus,

$$10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4},\ 10^{-5},\ 10^{-6},\ \&c.;$$

and the point (.) which is put after the unit's place of the former part of the series, must be put before the whole of this one. The above, collected into one expression, is,

.111111, &c.;

and resolving it into as many parts as there are terms or places in it, as we did in the case of the other, it becomes—

.1, .01, .001, .0001, .00001, .000001, &c.

10^0 is the number 1, or it is 1 neither multiplied nor divided; 10^{-1} is one-tenth of 1, or 1 divided by 10; 10^{-2} is one hundredth part of 1, or 1 divided by 100; and so of the others.

We thus obtain, by means of the decimal scale, three kinds of numbers, which are still all simple or abstract numbers, equally applicable to all quantities, of whatever kind they may be.

First, INTEGER NUMBERS, or *whole* numbers, the smallest of which is 1; and which, in the scale, read, from right to left, units, tens, hundreds, thousands, and so on. The different places in these numbers are found by multiplying the unit as often by ten as the number expressed by the exponent, they therefore express 1 and all numbers greater than, whatever may be their amount.

Secondly, DECIMAL NUMBERS, the largest of which is never exactly as much as the integer number 1, though it may approach nearer to this than any difference which we can name. These numbers have, that is, require to have, the distinguishing point on the left of them; they are read from the left to the right, tenths, hundredths, thousandths, and so on; and the values of the places or terms are found by dividing 1 by 10 as often as the number of the exponent. By means of these decimal numbers we can express all values which are less than 1; and though, as we can better explain in a future section, we cannot do this with perfect exactness except in particular cases, yet we can in all cases approach or approximate the value so nearly as that the difference shall be less than any thing that can be named.

It may be proper to give an example in illustration here. Well, the earth which we inhabit is a very large piece of matter; but if we take a tenth part of it, a tenth of that, a tenth of the result again, and continue this dividing by 10 for a sufficient number of times, we shall at last arrive at a quantity not only less than the smallest grain of sand, but less than any quantity which can result from the continued division of this grain by 10; for, if we name any quantity we must stop at some number of times dividing by 10; and there is nothing to prevent us from taking this quantity and dividing it again by 10 as often as we please. It is true that, in the case of the earth, we cannot perform even one of these divisions, but it is the beauty and advantage of arithmetic that we can count by means of the representations of numbers, instead of the things which we have occasion actually to number.

That this is a very great advantage we may see by the following case:—suppose that we had no arithmetic beyond simply counting one, two, three, and so on (and even this, as we have already seen, is not attainable without science), and that it were required to find the whole price of any number of pounds of goods at one shilling, one penny, and one farthing for each pound. The only way that we could go about even this very simple case would be to lay out the goods in single pounds in a row, place one shilling, one penny, and one farthing against every single pound, and then count the money thus placed.

Decimal numbers, or, as they are shortly named, "decimals," are merely a continuation of the very same scale as integer numbers; and by merely shifting the decimal point we may express the multiplication or the division of any number by 10 as often as may be necessary. Thus, in the following numbers, though each contains the very same figures or characters, in the same order, yet if we take them from the first to the last, each is one-tenth part of the one above it; and if we take them in the opposite order, or from the last to the first, each is ten times the one below it:—

879456321.
87945632.1
8794563.21
879456.321
87945.6321
8794.56321
879.456321
87.9456321
8.79456321
.879456321

The first of these numbers is read eight hundred and seventy-nine millions, four hundred and fifty-six thousand, three hundred and twenty-one; its exponent is ^{8}, because there are eight figures besides the units. There are nine divisions by 10 in the succeeding lines, and therefore the exponent of the last line is 9 less than 8, or $^{-1}$. The last line is read eight hundred and seventy-nine millions, four hundred and fifty-six thousand, three hundred and twenty-one, thousand millionth parts. The exponent of the last figure of the first line is ^{8}, that of the last figure is ^{0}; the exponent of the first figure of the last line is $^{-1}$, and that of the first figure of the same is $^{-9}$; thus there are 18 different exponents, answering to the 18 different places of figures.

We may farther remark, that every number expressed by the common figures or characters of arithmetic, may be considered as expressing a number of times 1 of its right hand figure; and that any number of 0s on the left of an integer, or on the right of a decimal number, do not in the least affect the value of the figures, or that of the number itself.

Thirdly, EXPONENTIAL NUMBERS.—These are altogether different in their nature from integer and decimal numbers; for, while both of these stand for numbers, or numbers of *things*, according as they are applied, exponential numbers stand for numbers of *times* multiplying or dividing, and never can be made to stand for numbers of things, or to be in any way expressive of the value of real quantities or existences. They are thus not numbers, but expressions for the relations of numbers to some one particular number; and this number, in our scale of arithmetic, is the number 10. If the exponent has not the sign — before it, the number of which it is the exponent always contains integers, and always one place more of them than the number of times 1 in the exponent. If the exponent has the sign — the corresponding number never contains any integers, but is wholly decimal, and there are always as many 0s on the left, or between it and the decimal point, as the number of times 1 in the exponent, wanting one. Thus, if we take the eighteen places of the first and last lines of numbers in the preceding example, with 1 in place of each of the figures, and mark the exponent of each, we shall have the following expression:—

$$10^8, 10^7, 10^6, 10^5, 10^4, 10^3, 10^2, 10^1, 10^0. 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}.$$

It will be seen, from this expression,—which goes as far both ways as there is ever much occasion for in practice, but which may be extended at pleasure both ways, by adding 1 to every succeeding exponent both on the left and the right, and taking care to continue the sign — before those on the right; that, read from right to left, this is a regular series of multiplications by 10; but that if we read it from left to right, it is a regular series of divisions by 10; and, as 1 in the exponent when it has not the sign — answers to one multiplication by 10, and when it has the sign — to one division by 10, it follows that there is no common addition or subtraction of those exponential numbers, for the addition of them is evidently the same as the multiplication of the numbers of which they are the exponents, and the subtraction of them is the same as the division of those numbers.

Farther, 0 in these exponential numbers means 1, and not nothing, as it means in common numbers; and the exponents which have the sign — before them do not mean imaginary numbers, that is, numbers less than nothing, they mean numbers of times divided by 10.

It is necessary to pay particular attention to the difference in meaning between those exponential numbers and common numbers, as used in ordinary arithmetic; because, though they have exactly the same forms, their meanings are altogether different. Exponential numbers are called LOGARITHMS, which means "the voices of numbers," that is, what they express, or the account which they can give of themselves; and this expression is always the number of times which 10 requires to be multiplied by itself, or divided by itself, in order to produce the common or natural number answering to the logarithm.

Those logarithms, or voices of numbers, are of vast use in many of the more elaborate parts of mathematical science, both in the investigation of principles and in the application of those principles to practical cases. But it requires more general views than any upon which we have hitherto entered, fully to explain even as much of their nature as is necessary for popular purposes; and therefore we shall need to revert to them in a future section, after we are in possession of the other elements which are necessary. We shall only add here, that by means of logarithms, calculations which required days before this invention, can be performed in minutes in consequence of it, and that they have enabled us to perform many calculations with ease which without their aid were altogether impossible. We have deemed it necessary to give the general definition, and also some short explanation of the nature of those exponential or logarithmic numbers along with the explanation of the notation and scale of the natural numbers; because when the meanings of the natural numbers are once rooted in the mind without any explanation, it becomes somewhat difficult to convey a clear and distinct notion of the same characters used as exponents.

SECTION IV.

COMMON OPERATIONS IN ARITHMETIC.

The use of arithmetic, and indeed of all branches of mathematics, consists in enabling us to find that which we wish to know but do not, and to do this by means of that which is already known to us; and the process by which this is obtained is called an *operation*. Or we may say that an operation is any process by which we are enabled, from known quantities, to arrive at the knowledge of quantities which are not known. In order to do this in any case we must have always one known quantity of the same kind with that unknown one which we are to find by the operation; and in arithmetical operations we must have this known quantity expressed in the very same unit of measure, or denomination, as the one whose value we seek. Thus if our object is to ascertain how many pounds will require to be paid under conditions which are given, we must have a pound, or something expressible in terms of a pound, among the data which we are to use in our operation; and in like manner, if we seek for the value of any quantity whatever, we must have either the unit in which that quantity is to be expressed, or something convertible into this unit, among the data. Thus if length of time were the quantity sought, we could not find it unless a quantity expressing time were given; and the same in all other cases. It is not necessary, however, that the given quantity should be in the same denomination with that which is sought, provided we know the relation between them. For instance, if a certain number of pounds sterling were the given quantity, and a number of French francs the quantity sought, we could find it with little less labour than if francs had been given, provided we knew the relation between a pound and a franc, or the number of any one of the two species of coin which is equal to a known number of the other species.

In the simple operations of common arithmetic this necessity for having among the data a representative of the quantity sought, seems so obvious a matter, as not to require being stated. But when we come to the more intricate parts of mathematics, and especially when we come to mixed problems—or things to be done—in which mathematics form only a part, it becomes a consideration of considerable difficulty as well as importance. The difficulty increases, too, in proportion as the mathematical part becomes smaller in respect of the whole case; and thus it is of great importance not in matters of calculation only, but in all matters generally, to make ourselves sure at the outset, that the data, or terms and conditions, by means of which we attempt to arrive at any result, contain elements sufficient for determining that result. We must bear in mind that any one of the conditions which are involved in the data by means of which we endeavour to find an unknown quantity may become the unknown quantity in a problem of another description; and that thus not only all sorts of quantities, but all sorts of relations of quantity to quantity, may become that element among the data, which is the virtual representative of the result; and we may add, that there are many cases in which the relations of quantities are not only all that we can obtain, but that those relations are often indeterminate, or inexpressible by our ordinary means of notation. This is not the place for entering into any particular explanation of when we have or have not data sufficient for the obtaining of the result which we seek; but we mention the subject at the outset, because it is one of great importance, and one which everybody who wishes to study mathematics easily and profitably must bear constantly in mind. To use a homely expression, the representative of the quantity sought among the data is of nearly the same importance as money in the purse is to a man who would buy a horse at a ready money market. The man may know exactly the place and hour of the sale, may have the means of arriving there in perfect time, may know who has got to sell exactly the horse which he wishes, he may have the blacksmith ready to put new shoes on the horse, and saddle, bridle, and portmanteau all in perfect order for a journey; but not one mile of that journey shall he ride if he is without the cash, the real element which represents the horse: and it is even so in all cases, whether mathematical or not.

In the arithmetic of simple or abstract numbers, viewed in its most elementary form, there are only four general problems, or kinds of results. The first is to find the *sum* of two or more numbers, and that is nothing more than finding one number which shall contain the number 1, the standard by which we measure all simple numbers, as often as it is contained in all the numbers whose sum is sought. The second is to find the difference between two numbers; and this difference is nothing more than a number which added to the less of the two given ones would make a sum equal to the greater, or which taken away from the greater would leave a *remainder* equal to the less.

The process by which we find a sum is called ADDITION, and that by which we find a difference is called SUBTRACTION. We shall very shortly notice the leading principles of these operations, before we proceed to the two remaining ones; because addition and subtraction are in some respects, though not altogether, the converse, or opposite each other. The operation of adding, and that of subtracting, are exactly the reverse of each other; but the result of addition is not exactly the opposite of that of subtraction in all cases, for the result of an addition must be equal to the whole of all the given quantities, and therefore it must be greater or less according as they, taken in their whole amount, are greater or less; whereas the result of a subtraction expresses merely the difference of the quantities, and thus it has no necessary reference to the entire value of the one or the other, or of both, but merely to the difference, or how much the one is greater and the other less. This consideration is worthy of some attention, simple as it is, and it will readily be understood when we consider that the difference of any two numbers, however large, that have all their figures the same with each other except the units, is exactly the same as the difference of these unit figures, and that if their difference is 1, then the difference between the numbers is exactly the same as that between 0 and 1. Thus the difference between 31587926 and 31587925 is equal to the difference between 0 and 1, that is, it is 1.

From this it follows that, when our object is to discover the difference of two quantities, we may take away as much of them as ever we please, if we take exactly the same from each; and that we may add as much to them as ever we please, if we add exactly the same to each. The first part of this very obvious power that we have over them is often of great use when we seek the difference of complicated quantities which contain many elements; and the second is the foundation of that borrowing and paying in common subtraction which is not unfrequently an unexplained puzzle to beginners in the arithmetical art.

The simple process of adding can hardly be made plainer by any explanation, and taken in detail it is not a difficult one, as we never have to add more than 9 at any one step; and therefore the only considerations in simple addition are, to take care that we do not call quantities equal, which are not so according to our scale of numbers, and then to express the sum in terms of the scale; or it may simply be expressed thus:—figures equally distant from the unit's place, or from the decimal point, whether in integers or in decimals, are quantities of the same kind, and can be directly added together; but figures situated at different distances from the unit's place or the decimal point, are not quantities of the same kind, and consequently cannot be directly added together. Farther, when the sum of any column of figures, that is, any row of them, equally distant from the unit's place, amounts to two figures in the common way of expressing numbers, the right hand one, or unit's figure of these, considered as a separate number, belongs to the same place as the column added, while the ten's figure belongs to the next place on the left, and must be carried there, and added in with whatever figures may be, or written down there if there are none. This is so simple as hardly to admit of explanation, for we have only to recollect that tens of units are tens, tens of tens are hundreds, and generally that 10 in any one place of the scale is always exactly equal to 1 in the place next on the left. It is usual to arrange the numbers with like places under each other, before proceeding to sum them; but this is a matter of convenience, not of necessity; and it gives considerable facility in calculation to practice the adding of numbers, when written the one after the other in the usual manner of writing.

The simple act of subtracting, like that of adding, can hardly be made plainer by words: the *rationale* of it consists in either beginning at the larger number, and counting downward to the less, or beginning at the less number, and counting upward to the greater; and the number counted, which would be the same both ways in the case of the same numbers, will be the difference. Thus, the difference between 16 and 9 is 7; for if we begin at the end of 16, and count down to 9, we have 7 to count; and if we begin at 9, and count up to 16, we have also 7 to count.

In subtracting simple or abstract numbers, we can actually subtract only the less from the greater; and if they consist of several figures, we must subtract from each other those places which are equally distant from the unit's place or the decimal point. Of course, in integer numbers, or numbers consisting partly of integers, or partly of decimals, the greater number is always that which contains the greater number of integer places; and of two numbers which have the same number of integer places, that which has the greater left-hand figure is the greater. If the numbers consist of decimals as well as integers, the greater is determined by the integer places, exactly as above stated; and if they consist wholly of decimal places, the greater is that of which the significant figures begin nearest the point; and if they begin at the same distance from the point, the greater is that which has the greater first figure. This depends upon the necessary and obvious conclusion from the nature of the scale of numbers, that 1, in any place of the scale, is greater than all figures which can possibly be written to the right of that place.

But although the whole of the greater number and the whole of the less are thus easily discovered, that is, though it is easily seen which is greatest and which least as a whole, it may happen that all the figures of the greater number, except the left hand one, are less than the corresponding figures of the smaller number; and that thus, though the whole of the one will, when taken from the whole of the other, leave a remainder, yet that the differences of the individual figures appear to be the other way. Thus 1000 is a greater number than 789, because it contains a figure more; but 9, 8, and 7 are all respectively greater than the figures which occupy the same places in 1000. But it will be recollected, that, if we add 10 to any figure of a number, and one to the next left-hand figure of any other number, we add the same thing to both numbers, and consequently do not alter their difference. Hence, whenever the figure in the larger number is the less of the two, we call it 10 more than it really is; and when we come to the next on the left of the smaller number, we call it 1 more than it really is, and thus we obtain the difference with perfect accuracy. The difference of the above numbers, taken this way, is 211.

In the subtracting of one number from another, or, which is the same thing, finding the difference of two numbers, we never, however large the numbers may be, have occasion at any one step of the operation, to subtract a number larger than 10, or to subtract it from a number larger than 19; so that the largest difference we have to count at any one time is 9, and the average is about 5, which is a very simple matter. But we may subtract a considerable number of lines of numbers, with nearly the same facility as one line; and in doing this we may have occasion to add several 10's to the figure of the larger number. It is of no consequence, however, how many there may be, if we compensate them by the adding of the same number of 1's to the next place to the left of the numbers which we are subtracting.

When numbers which are added or subtracted contain decimals, there are always as many places of decimals in the sum or the difference, as there are in that one of the given numbers which has the most, except in cases in which the adding or the subtracting changes some of the figures on the right hand of the sum or the difference into 0's, and then the result is shortened by as many figures as there are of these 0's.

There still remain two simple operations in Arithmetic, of which we shall now very shortly consider the principles; and these, of course, form the third and fourth in the order in which we have taken them.

Multiplication (literally, "*many-folding*,") is the process by which we find the *product* of two numbers, that is, the result which arises from repeating the one of them as often as is expressed by the other. If both are simple numbers, that is, numbers to which no particular values, expressive of real existences, are attached, it is of no consequence in what order they are taken; but in the case of real quantities there is a distinction, and a very important one. In all cases the two numbers are called the *factors* (the workers or producers) of the result or product; and in the proper understanding of them, the one factor, which is called the *multiplicand*, is a *number*, and the other, which is called the *multiplier*, is a *number of times*. As we have already said, the distinction between these is of no practical importance in the case of simple or abstract numbers, because they do not, in any case, express anything which has a real existence; but still the difference between a *multiplicand* and a *multiplier* is worthy of being borne in mind, even in them, because, if we do not attend to the distinctions of things in the most simple cases, they are sure to embarrass us in the more complicated ones.

We are, therefore, to understand, in all cases of multiplication, either that the multiplier is simply an expression for a number of times, and the product, after it is obtained, is of the same kind, and in the same denomination, or equal, unit for unit, with the multiplicand; or that the factors are both real quantities, and that the product is not any number of times either of them, but a new quantity, of a kind different from both.

There are only two common cases applicable to real quantities in which the product thus becomes a new quantity, altogether different in its nature from either of the factors; but those cases are so important, that we require to bear them in mind in all our reasonings about multiplications and products; we shall, therefore, very shortly advert to them here.

In the first place, every body who has at all thought on the subject is aware that, if we wish to know how much measure is in the surface of anything,—say of a rectangular table, that is, a four-sided table, the opposite sides of which are equal in length, and which also measures the same both ways from corner to corner in the diagonal direction—one takes the length and the breadth, both in the same measure, and multiplies them together; the result of which multiplication is the surface of the table in squares of that measure in which the length and breadth were taken. Thus, if the table in question were five feet *long*, and four feet *broad*, it is easy to see that the surface of it would contain twenty square feet, and that it would be of no consequence whether we called this twenty square feet, four times five, or five times four. But nobody will pretend to say that this twenty square feet is either four times the length, or five times the breadth of the table; for length and breadth are merely lines, and though we were to multiply either of them by the largest number that could be imagined, the product would still be a line; for there is no more surface in a mile, or a thousand millions of miles of mere line, in what direction soever it may lie, than there is in the ten thousandth part of an inch, or in any other line, however short.

The product, in this case, is not, therefore, of the same kind with either of the factors; for neither of these, however it might be multiplied, would produce the product, or anything which could be a part of it, however small. Neither are the two factors quantities of the same kind, though they are expressed in the same denomination, and obtained by the same kind of measurement.

It is true that each of them, taken singly and without reference to the other, is a measure of length, that is, the measure of a line; but in this sense neither of them can be a multiplier, in as much as there is no meaning whatever in any such expression as "a foot times," or "an inch times." If we were to suppose that both the four feet and the five feet, to which we have alluded in this instance, mean either length or breadth, both meaning the same, we could not, in terms of them, find the content or area of any surface whatever; for if it were said that the table is five feet long, and also other four feet long, or that it is five feet broad, and also four feet broad, all the conclusion to which we could possibly come from such data would be, that the length of the table was nine feet in the one case, and the breadth of it nine feet in the other. That the product of these two numbers of feet represents a quantity having any meaning at all, is, therefore, owing to the fact of the two lines of which the measures are given standing to each other in the geometrical relation of length and breadth; and the product of no two measures of lines ever actually expresses a surface, or otherwise has a definite meaning as a real quantity, unless the lines stand to each other in this geometrical relation.

If two measures of lines which stand in the relation of length and breadth, are both expressed in the same unit or denomination, the product always represents squares having all their sides equal to that denomination; but if they are in different denominations, the product expresses rectangles as long as the greater denomination, and as broad as the less.

The other case requiring particular notice is that in which there are three measures of lines which stand to each other in the geometrical relations of length, breadth, and thickness. The product of any two of those represents a surface of which these two are the length and breadth. Thus, for instance, if a brick is 9 inches long, 4 inches broad, and 2 inches thick, the content or solidity is the product of 9, 4, and 2, or 72 solid inches. Such a solid, which has its opposite faces and edges all equal to each other, and all its angles or corners equal, is called a parallelopiped, or rectangular prism. If the length, breadth, and thickness, which are called the three *dimensions*, are all different, as we have supposed them to be in the case of the brick; the six faces consist of three pairs, which are all different, but the two that form each pair are equal, and opposite to each other. Thus, if the brick is laid on one of its largest faces, as it usually is in building, the top and bottom are rectangles, expressible by the product of the length and breadth, or they contain each 36 square inches in the example which we have taken; the two sides are each the rectangle of the length and thickness, or 18 square inches in our example; and the two ends are each the rectangle of the breadth and thickness, or 8 square inches in our example.

Thus, in arriving at the content of the solid, we have three distinct kinds of quantities in succession; first, the original multiplicand, which is a line, and it is of no consequence to the ultimate result which of the three dimensions we use for this original multiplicand; secondly, we have a surface, as the result of the multiplication of this by another of the lines, and it is of no consequence which of the remaining two we use for this first multiplier; thirdly, we have the solid as the result of the second multiplication, and in the multiplier for it we have no choice, as the other two dimensions have been used in the previous part of the operation. If, however, these three measures of lines did not stand to each other in the geometrical relations of length, breadth, and thickness, the product would not be a solid, or a quantity having any meaning.

The numbers, of which the product representing the solid is composed, may have three distinct forms or values: first, if all the lines are taken in the same unit or denomination, the product will be cubes of that denomination; secondly, if two of them are in the same, but the third different, every 1 in the product will be a square of that denomination which occurs twice, the one way, and the third or different denomination, the other way; and, thirdly, if all the measures are in different denominations, the product will consist of solids which have their three dimensions respectively equal to their three denominations.

Though these correspondences between the product of two dimensions and a surface, and between the product of three dimensions and a solid, are very simple, and also very obvious matters, yet it is of importance to attend carefully to them, and notice the difference which there is between the result in the case of either of them, and, in any case of common multiplication considered in merely an arithmetical point of view. A clear understanding of this becomes of the more importance when we consider that this is the means by which Geometry and Arithmetic are connected together; and also, that the truths of Geometry, how beautiful soever they may be in themselves, are of no practical use, unless we can apply Arithmetic to them.

We have here three distinct kinds of arithmetical numbers, answering to three equally distinct kinds of geometrical quantities. First, we have simple or original numbers, which are not the results of any arithmetical operation, and they answer to lines; secondly, we have products of two factors, which answer to surfaces; and, thirdly, we have products of three factors, which answer to solids. Though the fact of these products, being really surfaces or solids, depends on the geometrical relation of the lines which the factors express, yet the products themselves have exactly the same relations to the factors, and are found in exactly the same manner, whether the factors have the geometrical relation to each other or not. Therefore, we may consider every product of two factors as representing a surface, and every product of three factors as representing a solid.

It only remains to be shown how the product of two factors is obtained, and what advantage we derive from the scale of numbers in the performing of this operation. Here it is not difficult to perceive and to understand that, on account of the analogy or resemblance which there is between a product and a geometrical relation, the geometrical element of the scale of numbers—the exponents or indices of the different terms or places—must be of far more use in multiplication, than it is either in addition or in subtraction; and upon this principle we may resolve the operation of multiplying into two parts—the particular product of the figures, and the exponent of that product, or the place which it should have in the scale of numbers. Both of these are necessary for determining the value of the product; and we shall see that it is of considerable use to us to be able to consider them separately.

If we had no way of considering the factors but as wholes, the multiplication of even small numbers would be a very laborious matter. To multiply 6285 by 1000, would, in this way, consist of counting 6285 a thousand times over; and this, simple as it appears to be, would take a man just about four years, counting 12 hours every day, at the rate of one number per second; but by Arithmetic it is done as fast as seven characters can be written; for we have, according to the explanation given of the scale, only to make the 6285 thousands instead of units; and this is done by adding three 0s on the right, thus, 6285000. We are much accustomed to hear boastings of the extent to which labour has been abridged by many of the mechanical contrivances of modern times, and in most cases these boasts are well founded; but even the best of them —the steam engine itself, which is "the Lion" by way of eminence—is absolutely nothing in power of abridgment to the scale of numbers in Arithmetic; and when we consider that *calculation* is one of the essential elements, without which not one of those mechanical contrivances could by possibility have been arrived at, we must be careful not to lavish all that praise and wonder upon the fruit, the greater part of which is, in truth, due to the tree.

From the nature of the scale of numbers, every number greater than 9 consists, in reality, of two or more numbers; and every number consists of as many distinct numbers as there are figures in it. It is of no consequence though some of the right hand or the intermediate places should be occupied by 0; for 0, when there are figures to the left of it in an integer number, is an expression for ten times; and when there are figures to the right of it in a decimal number, it is an expression for one-tenth, or division by ten.

In consequence of the advantages which we derive from the scale of numbers, the largest single product that we need ever to make use of, is 9 times 9; and this and all products of smaller numbers are usually committed to memory from tables in which they are inserted. But in this elementary part of the business it is much better to learn these products by actual counting, because it is more satisfactory, more convincing, and more likely to be remembered with little trouble. Matters which are committed to memory by mere drudgery, are rarely, if ever, understood; and therefore it would be an excellent rule for the elementary schools, to confine this labour to those subjects which are not worth understanding, or not intended to be understood.

When all the products of numbers under 10 have been learned, the next step of the business is to apply the principle of the exponents; and this principle is abundantly easy, because the addition of the exponents, as has been already shown, answers to the multiplication of the numbers; therefore, if both figures are integers, or both decimals, the distance of the product, or of its right-hand figure if it consists of two, from the unit's place, must be equal to the sum of those of the figures from which it arises; but if the one figure is an integer, and the other a decimal, then the distance of the right-hand figure of the product from the unit's place will be the difference of their distances, and on the same side with the greater, the numerical product being exactly the same, whether the figures multiplied are both integers, both decimals, or the one an integer and the other a decimal.

Also, each particular product of two figures is perfect in itself, and independent of the others. As the whole of any quantity is equal to all the parts, it of course follows, that, when all the figures of the one of two factors have been multiplied by all the figures of the other, and the results collected into one sum by addition, this sum is the true product of the whole of the one factor by the whole of the other, in the same way as if it were obtained in one original amount by the process of counting, without the aid of any scale of numbers, which, as has been already mentioned, would be so exceedingly tedious even in the case of numbers which are not very large.

If the factors in multiplication consist wholly of integers, then the product also is wholly integers, and the right hand figure of it, whether it happens to be 0 or anything else, is units; but if there are decimals in either or both of the factors, the product must obviously contain as many places of decimals as there are in them both, though it may happen that one or more of the right hand figures are changed into 0s by the multiplication. If both numbers, or any of them, be wholly decimal, it sometimes happens that the whole product does not contain so many figures as there are decimal places in the factors; and when this happens to be the case, the deficiency must be supplied by prefixing between the product and the decimal point as many 0's as shall make the number of decimal places equal to that in both factors.

As our present object is not to teach the practice of arithmetic, but to explain the simplest elementary principles, and to explain these only so far as that the advantage of the scale of numbers in the simplest operations may be seen, we shall only further remark on the operation of multiplying, that that operation may make the product greater than the multiplicand, or equal to it, or less, according to circumstances. If the multiplier is greater than 1, the product must be greater than the multiplicand; if the multiplier is less than 1, the product must be less than the multiplicand; and if the multiplier is equal to 1, the product must be equal to the multiplicand. This follows from the definition of multiplying, which means repeating a quantity as often as is expressed by another. From this it will be seen that in every case of multiplication the product must bear the same proportion, or ratio, or relation (for in mathematics all these words have nearly the same meaning,) to the multiplicand that the multiplier bears to the number 1. This principle is one of very extensive and important application; but we shall be better able to explain it and understand its use when we come to the consideration of quantities generally, or unfettered by arithmetical expressions. We shall therefore proceed very shortly to notice the fourth arithmetical operation.

DIVISION (literally "*seeing as two*, or, *as parted.*")—The common notion that we have of the operation of dividing, is that of separating a quantity into two or more parts; and hence the problem of which arithmetical division is the solution is often enunciated in words similar in meaning to these:—"To find any proposed part of a given number." But this definition is not general enough, because we always consider a *part* as being *less than the whole*, while the number which we find by division may be less than the given number, equal to it, or greater, according to circumstances.

The real problem is, "to express one quantity in terms of another, this other being considered as *one*, whether it be arithmetically expressed by the number 1 or by any other number."

All simple or abstract numbers, whether integers or decimals, are expressed in terms of the number 1, as they are all brought to the unit's place, by adding 0s to integers and prefixing them to decimals when necessary. The number which expresses the one of two numbers in terms of the other must also be expressed in terms of the number 1, otherwise its value would not be known according to that standard by which we measure the values of all simple numbers. Therefore the general problem in division resolves itself into this: "To find a number which shall have the same relation, or ratio, or proportion, to the number 1, which the one of two given numbers has to the other."

The number in terms of which the other is to be expressed is enabled the *divisor*, which means the "divider," the measure or instrument of division; the number which is to be expressed in terms of the divisor is called the *dividend*, which means the "divided," the subject of the division; and the resulting number, which expresses the dividend in terms of the divisor, is called the *quotient*, which means "as much, or as many, as there shall be."

This last is a vague or indeterminate expression; and that it should be so is necessary in order to include all cases. The reason of this will become apparent when we consider that two conditions are required in this quotient, and these two conditions may or may not agree exactly with each other, according to the particular case. The quotient must express the dividend in terms of the divisor; and this is the quantity actually sought, so that no deviation from it can be admitted. But again, the quotient must be expressed in terms of the number 1, in order that it may come within the limits of common arithmetic, and be of use in practice. Now, though these two modes of expression are in many cases perfectly compatible; yet any one will readily see that it is not necessary they should be so; for, the number 1 is a fixed and invariable standard in the case of abstract or simple numbers,—though it does not represent *one* of any particular kind of *quantities*, but stands ready to express *all ones*, of what kind soever they may be, and whether in themselves of greater value or of less. A divisor, on the other hand, may be any thing: it may be 1, a number of times 1, less than 1, an expression which is not 1, or a nameable number of times or part of 1; and in the last case the quotient cannot of course be expressed in terms both of 1 and of the divisor.

A very simple instance will illustrate this: the number 10, as it is written, is expressed in terms of the number 1 only, because, according to our common notation, it expresses ten 1's. Now, suppose that it were required to express this 10 in terms also of the number 2, it is evident that the expression would be 5, because 5 times 2 and 10 times 1 are exactly the same. The number 10 is therefore capable of being expressed both in terms of 2 and 1. But let us endeavour to express it in terms of the number 3, and we find that the quotient is not expressible in terms of the number 1; the nearest expression which we can find is 3, but three 3's make only 9, and we want a number of 3's that shall make 10; 4 will not do, because four 3's make 12, which is 2 more than 10. Therefore, in integer numbers, all that we can arrive at is three 3's, and 1 over or remaining. But we can get some more of the expression in decimals, for upon the same principle that 1 ten is 10 ones, 1 unit is 10 tenths, and the result of this is 3 and 1 tenth over: 1 tenth is 10 hundredth parts, so that we again have 3 hundredth parts and 1 over; and if we continue to ever so many places of decimals we shall always have the same number 3 and 1 over; so that the number which expresses 10 in terms of 3 cannot be exactly expressed in terms also of the number 1, so as to be brought within the scale of arithmetic, either in integers or in decimals. To whatever length we carry it, it is 3.33333 and 1 of the last place or term divided by 3, or 10 of another place more to the right, remaining to be expressed in terms of the number 3, and so on—without end.

We shall have occasion afterwards to point out the circumstances which determine when a dividend can be expressed in terms both of the divisor and the number 1, and when it cannot; and all that we require to know in the mean time is, that this expression is sometimes possible and sometimes not.

With this understanding, that the dividend is sometimes equal to the product of the divisor by a number which is expressible by common notation in terms of the number 1, and sometimes not, we may consider division as exactly the converse or opposite of multiplication; and the general problem into which it is resolvable to be, "given a product and one of the factors to find the other factor."

Simple as this problem appears to be, there is no way of solving it directly, with the consideration that both divisor and divident are wholes, but by trial; and even in a very simple case, the number of trials before the quotient, or even a near approximation to it, could be obtained, would be very great—so great, that success would be altogether hopeless.

But, as this is the most difficult of all the simple or elementary operations in arithmetic, so the scale of numbers comes more effectually to our aid than it does in any other. By means of it we are enabled to resolve that which we seek into two parts: first, the number of figures or places of which the quotient or number sought shall consist; and secondly, what these figures shall individually be. As no individual figure can be greater than 9, there cannot be more than about half that number of trials necessary, even for mere beginners; and a very little practice enables us to choose the proper figure without any trial. The rule for the number of figures in the quotient may be understood with equal ease: as many figures on the left of the dividend as will contain the divisor once and not more than nine times, will give the first or left hand figure of the quotient, and every additional figure which remains of the dividend will give one figure more. If there is a remainder after all the figures of the dividend have been used, the division may be continued decimally by adding 0's to the successive remainders, till it either terminates or is carried as far as may be thought necessary. If the dividend is less than the divisor, the quotient must evidently be less than 1, or consist wholly of decimals; because a smaller number does not make 1, in terms of a greater, as, for instance, 7 does not make 1 in terms of 8, and therefore the quotient of 7 divided by 8 can be expressed in the scale of numbers, or in terms of the number 1, only by a decimal number, the whole of which, of how many figures soever it may consist, is less than the number 1. When the quotient is thus wholly decimals, the first or left hand figure of it must evidently be as many places to the right of the decimal point as there are decimal places used in obtaining that figure.

Beginners sometimes find a little difficulty in the division of decimal numbers, not in obtaining the figures of the quotient, because these are found in exactly the same manner, whether the dividend and divisor are both integers, both decimals, or partly the one and partly the other, but in determining rightly the situation of the decimal point. This difficulty may be easily got rid of by attending to the following considerations:— Every number, whatever it may be, expresses a number of its right hand figure; thus 3500 expresses thirty-five hundreds, because 5, the right hand figure, is in the place of hundreds; 350 expresses thirty-five tens, because 5, the right hand figure. is in the place of tens; 3.5 expresses thirty-five tenths, because 5, the right hand figure, is in the place of tenths; .035 expresses thirty-five thousandth parts, because 5, the right hand figure, is in the place of thousandths; and the same of any other number, whatever number of figures it may consist of, and how far soever its right hand figure may be, integrally to the left, or decimally to the right, of the decimal point, always expresses as many of the place of that figure, as it would express of units, or of times the number 1, if its right hand figure were in the unit's place.

From this, it evidently follows that the arithmetical expression of two numbers according to the scale does not express the proportion of their real values unless their right hand figures occupy exactly the same place in the scale, that is, be at exactly the same distance from the decimal point, whether to the left hand of it or to the right.

We are in the habit of expressing all integer numbers in terms of the number 1, which is accomplished by simply supplying as many 0's as will bring the number down to the unit's place; thus one 0, if the number is wholly tens, two 0s if the number is wholly hundreds, three 0's if it is wholly thousands, and so on; and it is evident that every 0 which we thus add multiplies the numerical expression of the number by 10, that is, makes it ten times greater in number, but not any greater in its whole value; so that every 0 thus added makes each individual figure of the number only one-tenth part of what it was without the 0, or we have ten times as many as we had before; but each 1 of that number is ten times less than it was before, and thus the entire value is not in the least altered. Two 0s increase the number one hundred times, and make every 1 of which it is composed only one-hundredth part of what it was; and generally whatever number of 0's we annex to the right of a number, without altering the whole value of that number, we divide each individual 1 which the number contains as often by 10 as there are 0's.

By the application of this principle any number whatever may be expressed in terms of any lower place in the scale, whether integral or decimal, by annexing as many ciphers to the right of it as shall bring it down to the required place; and therefore, when we have a divisor and dividend which contain different numbers of decimal places, we have only to add to that which has fewest as many 0's as shall make its number equal to that in the other, and then divide the one by the other in exactly the same manner as if they were both wholly integers and the right hand figures of them units. Of course, whatever figures of the quotient are obtained from the divisor and dividend so prepared, must be integers, but if more 0's be annexed either to the dividend or to the remainders, there must be a decimal place in the quotient answering to each of them.

The principle which this involves is the most important one in the whole science of arithmetic; because it enables us to separate the absolute values even of numbers from the numerical values of them, and thus to consider their relations generally; and, it is the application of those general relations which is our chief element in the finding of unknown quantities by means of known ones; for, if we have a quantity of the same kind with that which we seek, and know the relation which this quantity bears to that which we seek, we are in a condition for finding the value of the unknown quantity, in terms of the known one. This part of the subject is called *the doctrine of* PROPORTION, that is, the doctrine of the relation which the magnitude, or value, of one quantity taken as a whole bears to that of another quantity taken as a whole. In order to have this relation, the quantities must be such that we can at once say that they are equal, or that one of them must be greater than another; and if they are expressed in numbers, those numbers must be expressed in terms of the same place in the scale of numbers; for if the arithmetical expression were not of this description, the numbers would express a different ratio from the quantities. A very simple instance will serve to illustrate this: there is a certain ratio or proportion between 7 sovereigns and 8 guineas; but it will be at once seen that this is not the ratio of the numbers 7 and 8, because 1 in the one of them is not equal to 1 in the other of them, for a sovereign is equal to 20 shillings and a guinea to 21 shillings; and therefore, before we can find two simple numbers which will express the ratio, we must turn both into their values in shillings, which is evidently done by multiplying the sovereigns by 20 and the guineas by 21, which numbers are 140 and 168. But if we look at the numbers which we multiply, we find there is 7 on the one side, and 21, or three 7's, on the other; and again, that there is 20 or five 4's on the one side, and 8 or two 4's on the other. We may therefore throw the 7 and the 4 out of both sides, and there remain 1 and 5 answering to the 7 sovereigns, and 2 and 3 answering to the 8 guineas; and if we take the products answering to each, we have 7 sovereigns to 8 guineas, in the ratio of 5 to 6, which is really a more simple ratio than that of the numbers.

Before proceeding to the explanation of this doctrine in such a manner as to have clear notions of its usefulness, and due expertness in the application of it, it becomes necessary to introduce principles more general than can be introduced by means of the common arithmetical figures; because we have seen that the numbers in which quantities are presented to us do not necessarily express the ratios of those quantities of which they are the arithmetical names; and also because, as we have already partially seen, from the impossibility of dividing some numbers exactly by some others—as we shall afterwards see more at length—all ratios are not expressible in terms of 1 in any place whatever of the arithmetical scale. We shall therefore only mention farther, that as the product in multiplication, when divided by any one of the two factors, must necessarily give the other factor as quotient, it follows that the quotient of a number accurately expressible by the scale, divided by one not accurately expressible, may be a number accurately expressible. We shall now proceed to the consideration of quantities generally, and without any reference to whether they can or cannot be expressed arithmetically by numbers.

SECTION V.

GENERAL OR ALGEBRAICAL EXPRESSION OF QUANTITIES.

It is a general rule in nature, in art, and in science, that everything which is peculiarly well adapted for the accomplishment of some one particular purpose, is, for that very reason, the worse adapted for every other purpose. This is remarkably the case with the scale and notation of numbers in our Arithmetic. It is difficult to imagine any means by which individual operations in numbers could be performed with so much ease and certainty; but the great facility and precision with which we are able to manage each particular case, become obstacles in our way when we attempt to investigate general principles. The whole expression—all the characters which are before our eyes, apply to the particular case only, and each number tells us nothing but its relation to the scale. There is no trace of the operation in the result, and very often there is not a vestige of the original or given numbers, by which the result has been obtained. Thus, for instance, 49 is the product of 7 by 7; and the product of no other two integer numbers, except of 49 and 1, which is not, properly speaking, a product at all. But there is nothing in 49, as it appears arithmetically, to let us know that it is the product of 7 by 7, or of any two numbers whatever; and any one who had not learned by rote the products of all numbers up to 9 times 9, or 81, would be just as likely to suppose that 47 were the product of two numbers; but we find by actual trial that 47 is not the product of any two numbers, each greater than the number 1. Even in the very simplest instances which we can take, we find no trace whatever either of the given numbers, or of the operation, in the result. 5 is the sum of 2 and 3; but there is no appearance in it, either of its being connected with 2 and 3, or of its being a sum at all. It is one simple and original mark, and leads us to think only of a number of times 1—of a short expression for the same number of dots (. . . . .).

An arithmetical expression does not, thus, give any account of itself, and an arithmetical operation has no story to tell. However long and complicated, however short and plain, however tedious from want of the knowledge of principles, or simple from the possession of this knowledge, such an operation may be, it offers no instruction to the ignorant; for though some of the numbers may appear as data, or as being originally given, and others as results; yet all that is done in order to obtain the results from the data is concealed; and, in order to understand the operation, the student must bring to it the very same degree of knowledge which was required for the original performing of it. A book might be filled with such operations, but it would be a book containing no knowledge; and as we cannot separate the principles from the particular numbers contained in the example, the only knowledge that we can obtain is a set of empirical rules, the truth of which we have no means of bringing to the test.

It is for these reasons, that the arithmetic which we usually learn at school is not only wholly useless to us as an instrument of knowledge, but acts as a barrier in our way when we attempt to understand anything of the other branches of mathematics. It is as if we were to sow flowers, and expect them to spring up, and grow, and produce a crop of plants. The principles of vegetable life are in the flowers, but they are not developed: the development is in the seed, and that seed we cannot obtain, if we separate the flower from the parent plant. Just so, when we are conversant with numbers only in the way of the common arithmetical operations, we are without the principles of knowledge in such a state as that one part may be fruitful of other knowledge.

There is this farther disadvantage in the application of empirical rules of which we do not comprehend the meaning and in the performing of operations the reason of which is a mystery to us, that we never enter heartily upon such subjects, and never at all, if we can avoid them. Now, as this repulsive matter meets us at the very threshold of reasoning, the mischief which it does to our powers of reasoning, upon all subjects, is incalculable; and as the investigation of mathematical truth is the only means by which the mind can acquire *the confidence of truth* in all matters of reasoning, it is not easy to calculate how much of the intellect of the world is condemned to lie useless by this cruelly absurd practice, which gives us the full measure of the labour without any of the advantages. We at once get rid of all these difficulties if we have recourse to *Algebra*, and so make ourselves masters of the principles, before we apply them arithmetically to particular cases. It is a very general opinion that there is something mysterious and difficult in Algebra, but no opinion can be more erroneous; it is as natural to the buoyancy of the youthful mind as running is to the vigour of young limbs, only the arithmetical notions (or rather want of all notions) act as a trammel upon us—our feet are tied before we are allowed to run.

ALGEBRA literally means "the *consolidation*"—that which sets the whole matter before our eyes in the clearest manner, and in the smallest possible compass. The name is very faithfully descriptive of the science; and surely being complete, being clear, and being brief, are not very like the usual causes of difficulty and mystery! Every quantity, every relation, and every operation, is fairly marked down; so that one line, or often one single expression consisting of only a few letters and marks, tells a longer story, and tells it more clearly, than if it occupied a whole volume in the common words of language. The end is thus seen from the beginning, and all the steps of the most complicated operation are shown at a glance, with a satisfaction to the mind which cannot be obtained by any other means.

As this is accomplished without any reference to the numerical values or expressions of quantities, all results which are arrived at become general, and are applicable to all quantities of the same kind which have the same relations to each other. Thus every algebraical operation is the investigation of a truth, and every result of such an operation becomes a rule for the solution of all arithmetical questions which depend on the knowledge of that truth.

The means by which all this is accomplished are so exceedingly simple, that perhaps nobody ever arrived at the knowledge of the more elementary ones, unincumbered by arithmetical considerations, without wondering why he should have required any teaching. There is only an alphabet to learn, and it is a very short one; and when we have once mastered this, if we understand any subject in itself, we can be at no loss in expressing and treating it algebraically. This alphabet forms what we may call

ALGEBRAICAL NOTATION.

There are two ways of viewing the same quantity: we may consider it simply as one whole, or we may consider it as in some way made up of parts, or as the result of some operation; and though the quantity may be in both cases exactly the same, yet, as in Algebra we must express all that we mean and all that we do, we must have the means of pointing out whether we view the quantity simply as a whole, or as a compound or result.

When we consider a quantity simply as a whole, we express it by the very shortest name or mark which we can possibly use, namely, a single letter of the alphabet—one of the letters near the beginning, as a, b, or c, if the quantity is a known one; and one near the end, as x, y, or z, if the quantity is unknown.

This use of the letters is purely arbitrary; but they are at least as good as any other marks, being simple and familiar to everybody. In English, the letter *a* is peculiarly happy as a general expression for quantity, because, as a word, it is the most general and indefinite in the language. *a* man, or *a* book, is the most general expression that we have for one man or book; and if we leave out the name, and use *a* alone, then it means any one thing, or circumstance, or relation whatsoever, which we can imagine to be a quantity.

The expressing of unknown quantities as well as known ones in Algebra, is a very great advantage, because it enables us to state completely those relations by means of which we arrive at the value of the unknown quantity in terms of known ones. When Algebra was first introduced into Europe, it was named, from this circumstance, *Cossics*, from the Italian word *cosa*, a thing—"the thing sought," being written among the data, as well as the things which are given.

Though the letters used in algebraical notation have no fixed numeral values, yet the same letter is never used for two different values in the same operation, but is understood to be the same at every step; or if it acquires a new value, which cannot be expressed by other quantities or relations, the new value must be expressed by an accent, or some other mark: thus, a for the first value, a' for the second, a'' for the third, and so on. This mode of expression is not required in elementary cases.

When a quantity is considered as compound, there must be a letter, or other expression, for each part of which it is composed; and, in addition to these, there must be some means of pointing out the relations in which the parts of the compound stand to each other.

The simplest form in which a compound quantity can exist, is that in which it is formed of two simple quantities; and there are four leading relations in which the very same two parts of a compound quantity can stand to each other, all of them giving different values to the compound quantity as a whole.

The two parts may, in all these cases, be represented by the same two letters, as a and b.

The first general relation is that of the addition of the two quantities, which is expressed by putting the sign + (*plus*) between the two letters, thus, $a + b$, which is read, "a plus b," and means the sum of the two quantities which are expressed by a and b.

The first quantity, a, which has no sign before it, is also understood to have the sign + ; and therefore it is of no consequence to the value of the whole expression whether we write $a + b$, or $b + a$; for it is evident that the sum of the same quantities is the same in whatever order we take the individual quantities. A pound, a crown, and a shilling make twenty-six shillings, whichever of them we take first, or second, or last.

Here it may not be improper to notice the method of representing the very simplest relation which the whole of one quantity can have to the whole of another, whether they be both simple, both compound, or the one compound and the other simple—namely, the relation of perfect equality. This is done by writing the sign =, which is read "equal," or "equal to," between the equal quantities: thus, $a + b = b + a$, expresses that the sum of a and b is equal to the sum of b and a. If we were to consider this sum as one whole and simple quantity, and represent it by a new letter, as by the letter s, then we might write $a + b = s$.

The second general relation of the two parts of a compound quantity to each other, as affecting the value of that compound quantity, is that in which the one of them is subtracted from the other. This is expressed by prefixing the sign — (*minus*) to the quantity to be subtracted. If b is the subtractible quantity, the expression is $a—b$; but if a is the subtractible quantity, then it is $b—a$. But these two expressions are not equal to each other, except in one particular case, that in which $a=b$, and then, both $a—b$ and $b—a$ are equal to 0. In every other case, the value of the one expression is greater than 0, and that of the other less; and the one is just as much greater than 0, as the other is less. If the smaller of the two quantities has the sign —, the value of the two will be greater than 0; and if the greater has the sign —, the value of the two will be as much less than 0 as it is greater in the other case.

Here it is necessary to mention the means of expressing another relation of the whole value of one quantity to the whole value of another; namely, the relation of *inequality*. But mere inequality is not a definite relation from which any useful conclusion can be drawn: for, as either of the two quantities may be the greater, we are left with two opposite meanings, both equally applicable, if we do not know which of the quantities is the greater; and so, if we have not the means of determining this, the fact of their being unequal can be of no use to us. But if we know which quantity is the greater, the relation of inequality becomes useful, whether we happen to know by how much it is greater or not; for the simple fact of being greater or less is all that we may have occasion for in some cases. Now as the relation of equality is, as already mentioned, expressed by writing between the quantities, the sign $=$, consisting of two lines, equally open or apart from each other at both ends, inequality is very naturally expressed by 7, which consists of two lines, open at the one end but meeting at the other; and the open end is turned to the greater quantity. Thus, if a is greater than b, we may express the fact by $a \, 7 \, b$, which is read, "a greater than b;" or we may turn it the other way, $b \angle a$, and read, "b less than a," which is the very same meaning differently expressed. If $a \urcorner b$, then it follows that $a-b \urcorner b-a$, or $b-a \angle a-b$.

Though + is the sign of Addition, and — the sign of Subtraction, they have a much more extended meaning than those operations have. The sum of two quantities may always be expressed by $a+b$, and the difference by $a-b$, if $a \urcorner b$, or by $b-a$, if $b \urcorner a$; but we can use the expression $a+b$, or $a-b$, in cases where arithmetic could not be applied. a and b may be *any* two quantities, whether of the same kind or of different kinds, and whether it be or be not possible to express them arithmetically.

We must be careful also not to confound this general or algebraical use of the signs + and —, with the particular use of them as applied to exponents, of which we gave some account when endeavouring to explain the scale of numbers. Thus, in the two expressions $a+b$ and a^{+b}, the sign + has very different meanings. We may remark, that in the expression a^{+b}, it is not necessary to write the sign + before the exponent ^{b}, any more than before the quantity a, for the position of ^{b} shows that it is an exponent not a simple quantity, and we never write + before the first character of any expression, that being always understood to leave +, unless — is prefixed to it.

Well, let us return to $a+b$ and a^{+b}, retaining the sign in the second quantity, which, though not necessary, does not alter the value, as the exponent is really + in all cases where it has not —. In order that we may get definite values, let us take a particular case, by using numbers for the letters; and as a and b are perfectly general, we may use any numbers for them. So let us call $a=5$, and $b=4$. Then $a+b$ will become $5+4$, and a^{+b} will be 5^{+4}, or without the sign 5^4.

Let us now find and compare the values of these expressions: $5+4$ is the sum of 5 and 4, which, by counting, we find to be 9; therefore, if $a=5$ and $b=4$, then $a+b=4+5=9$.

Now for the other quantity, 5^{+4} or 5^4. The 4 here is an exponent, and means the number of times that 1 is to be multiplied by 5. One multiplication gives us 5; a second, five times 5, or 25; a third, five times 25, or 125; and the fourth or last one, five times 125, or 625. So that if $a=5$ and $^{+b}=4$; then $a^{+b}=5^{+4}$ or $5^4=625$, which is a very different value from 9.

Let us now examine the case of the same expressions, in which the second quantity, b, has the sign $-$, $a-b$, and a^{-b}. In this case we cannot dispense with the sign $-$ before b in either expression: so let us use the same numbers as before, and get the values. If $a=5$ $b=4$, then $a-b=5-4=1$. So that the value of the first is 1.

Then for the second: 5^{-4}, from what was said of exponents, means that 1 is to be divided four times by 5. Once dividing gives us two tenths, or .2; a second time, .04; a third, .008; and a fourth, .0016; so that if $a=5$, $b=4$, then $a^{-b}=5^{-4}=.0016=$ sixteen ten thousandth parts of the number 1. If we express 1 and this number both in terms of 1 unit, we have $1=10000$ and $.0016=16$; and if we divide the greater by the less, we have 625; so that if $a=5$, $b=4$, then $a-b$, though equal only to 1, is 625 times as much as a^{-b}. There is another meaning of exponents, in which the sign $-$ indicates a different result; but this can be noticed with more advantage afterwards.

When we say $a-b$, the quantity expressed by b has exactly the same numerical amount, if it be such as that we can exhibit it in numbers, as when we say $a+b$; but it has a very different effect upon the whole value of the expression. We have seen that when $a=5$ and $b=4$, $a+b=9$, and $a-b=1$, the difference of which values, or $9-1$, is $=8$; and 8 is equal to twice b. But a is the same in both expressions, and therefore it does not affect the difference, so we may leave it out in both; and we have this general expression: the difference which the same quantity with the sign + and the sign — makes on the total value of any compound quantity of which it forms part, is equal to twice that quantity: hence, if we change the sign of any part of a quantity from — to +, we add twice that quantity to the whole; and if we change the sign from + to —, we take away twice that quantity from the whole.

+ has, thus, always an increasing or augmenting effect, to the full extent of the quantity of which it is the sign; and — has always a diminishing effect, to the full amount of the quantity of which it is the sign. They are thus the opposites of each other; and whatever is done by the writing of a quantity with the one sign is undone by writing the same or an equal quantity with the other. Thus $+b-b=0$; and $a+b-b=a$. Generally, any quantity, relation, or anything else that can be expressed with the one of these signs so as to have a meaning, has the very opposite meaning with the other sign. When this meaning is a real value which can be expressed by means of numbers, the two signs applied to the same quantity may be considered as lying on opposite sides of 0, and equally distant from it. 0 in this case, must not, however, be confounded with 0 considered as an exponent; for the exponential 0 means the quotient arising from the division of a quantity by itself, which is of course always equal to the integer number 1, while 0 considered with regard to the general value of quantities, really means *nothing*; and upon this principle, the sum of two equal quantities, one with the sign +, and the other with the sign —, is always equal to nothing.

Every part of a compound quantity which is separated from the other parts, either by the sign + or the sign —, is called a *term* of the quantity; and if they are different quantities, used as such, expressed by different letters, each has, of course, a different *name;* and the words, *binomial, trinomial*, and so on, may be used, according as the quantity is made up, of two, of three, or of more terms: $a+b$ or $a-b$ is a binomial; and as quantities are of considerable use, considered in this point of view, it is necessary to bear in mind what is meant by a binomial.

Though a quantity may be expressed in any number of terms, with the signs + and —, or either of the two between them, it is often necessary to consider it as one whole or simple quantity. This is expressed either by drawing a line over the whole of it, which line is called a *vinculum* or band, or by enclosing it within parenthesis;—thus $\overline{a+b}$, or $(a+b)$ means the sum of a and b considered as a whole quantity. The parenthesis is the preferable mode of expression; because these characters mark the beginning and end of the quantities which, taken altogether, are to be considered as one whole. This is a double method of expression, and points out at once the whole quantity, and the parts of which it is composed. It is, consequently, of considerable value in Algebraical notation.

It is necessary that every one who wishes to understand any thing of Algebra, or indeed of the *principles* of any kind of calculation, should have very clear notions of the nature and difference of the signs + and —; and it is for this reason that we have examined them somewhat more in detail than we shall be able to do many other parts of the science.

The third general relation of the parts of a compound quantity, is that in which they are *factors*, and the expression is their product. There are several ways of doing this: if the factors are single letters, their product may be expressed by writing the one after the other, without any sign between them; thus, $a\,b$ is the product of the quantities a and b. In a product, one of the factors, namely, the multiplier, is always a number; for the product means the multiplicand taken as many times as is expressed by the multiplier, though this number of times may or may not be expressible in terms of the number 1, or according to the common scale of numbers. But every thing, whether expressible by the common notation of arithmetic or not, is expressible by Algebra; and therefore while $3\,a$ or $4\,a$, is an arithmetical expression for one known number of times the quantity a, $n\,a$ may be considered as a general expression for any imaginable number of times the quantity a, whether that number can or cannot be expressed arithmetically.

But a quantity which is compound may be a factor in multiplication, as well as a quantity which is simple and as such expressed by one letter. Thus $a+b$, $a-b$, or $a\,b$, may be multiplied by any quantity whatever; and we require to have a means of expressing the multiplication in this case, without the performing of any operation. If we are to multiply a simple product, as $a\,b$, by another factor, for instance, c, we have only to join c to the others, without any sign, thus $a\,b\,c$ is the product of $a\,b$, by the third quantity c. If the multiplier were also a product, as for instance, $c\,d$, we would have only to joint them to the other letters; thus $a\,b\,c\,d$ expresses the product of $a\,b$ by $c\,d$. From this we can draw some inferences which are not unimportant in the practice of calculation,—namely, that, if we have to find the product of any number of factors, we may take them in any order that we please; and that to multiply either factor before multiplying, produces exactly the same result as multiplying the product after.

If one of the factors in a product consisting of only one term is a number, that number is usually called the numeral *coefficient* of the letters; that is, of the factor or factors which are expressed by letters; and even though all the factors of a simple product are letters, it is usual, when one of them expresses an unknown quantity and the others known ones, to consider the product of the known ones as the *coefficient* of the unknown. Thus, in the expression $3\,a$, 3 is the numeral coefficient of a; and in $a\,b\,x$, the product of a and b is the coefficient of x. We need hardly mention that a numeral coefficient is always definite, and means as many times the other part of the term as the number expresses; but that a coefficient consisting of letters is general, and does not mean any definite number of times, unless the letters stand for quantities of which the numeral values are known. Thus $a\,b\,x$, without further explanation, means any number of times whatever, the quantity represented by x; but if $a=5$, and $b=4$, then their product is 20, and $a\,b\,x=20\,x$. The doctrine of coefficients, as distinguished from the quantities of which they are the coefficients, is one of considerable importance; for, although the name "coefficient" means merely a "worker together" with the other factor or factors of the product, yet the coefficient always means a *multiplier*, or "number of times;" and the other factor, or their product, if there are more than one, means a *multiplicand*, or that which is multiplied; and when the term means a real quantity, that quantity is always of the same kind with the part which is not the coefficient, and of the same number with that which is.

There are other two considerations in the case of terms which are simple products, that is, which are affected only by one sign + or −, to which it is necessary to pay attention. These are, first, when the factors are all the same with each other, or are or may be expressed by the same letter; and secondly, the sign which the product shall have.

When the same letter occurs more than once as a factor in a term, the number of times that it occurs may be expressed by an exponent: a numeral exponent, if the number of times that the factor is repeated is known, and a literal exponent if the number is not known. Thus $a\,a\,a$, may be expressed by a^3; $b\,b\,b\,b$ by b^4; $a\,a\,b\,b\,b$, by $a^2\,b^3$, and the same in all other cases.

The sign of the product of two factors, whether those factors are expressed by the same letter, or by different letters, depends upon the signs which the factors have; but the algebraical expression of the product by letters, or the arithmetical expression of it if it admits of being arithmetically expressed, is the same for the same factors whatever their signs may be. Thus, the product of a by b, is always $a\,b$; and if they both have the sign + or, which is the same, are written without any sign, the product will have the sign +, or may, if it stand alone, or first in a quantity consisting of more than term, be written without any sign.

The reason of this is quite obvious; for the product of a and b, that is of $+a$ and $+b$, means that the one considered as a positive or real quantity is to be taken or repeated positively, as many times as the other expresses.

If the one has the sign + expressed or understood, and the other the sign —, the product must have the sign —; because, if we are to consider the +, or positive quantity, as the multiplicand, and the —, or negative quantity, as the multiplier, then the meaning of the product is, that the positive quantity shall be taken away as often as is expressed by the multiplier; and, on the other hand, if we consider the —, or negative quantity, as the multiplicand, and the +, or positive one, as the multiplier, then the negative quantity is to be added as often as is expressed by the positive multiplier; and the addition of a negative quantity is the same as the subtraction of an equal positive quantity. Thus, the product of a and $-b$, or of b and $-a$ is always $-a\,b$, whether the one or the other be the multiplier.

But if they both have the sign —, the product will have the sign +, and will be exactly the same as if they both had the sign +; that is, the product of $-a$ and $-b$ is exactly the same as the product of $+a$ and $+b$, or of a and b, that is, it must be $+a\,b$, or $a\,b$.

This case appears a little strange, and even inconsistent, to beginners; because the factors $-a$ and $-b$ are not only different from $+a$ and $+b$, or from a and b without any signs; but each of them with the sign — is twice itself less than it is with the sign +; $-a$ being equal to $a-2\,a$, and $-b$ to $b-2\,b$. Each of them with the sign — is just as much less than 0, or nothing, as it is greater than 0, or nothing, when it has the sign +.

At first sight it seems rather singular that two quantities, each of which is all that it expresses less than nothing, should have exactly the same real and positive product as if each of them were all that it expresses greater than nothing; and yet a very little consideration will convince us that such must be the case. If the multiplier is negative, or has the sign —, the multiplicand must be taken away as often as is expressed by the multiplier; and it depends on the nature of the multiplicand *what* is to be taken away this number of times. If the multiplier is positive, a positive quantity equal to the product must be taken away, that is the value must be diminished to the whole extent of the product; and if there is nothing besides this product, all that can be done is to mark it with the sign — as being its whole amount, or value, less than 0. But if the multiplicand is negative as well as the multiplier, then the multiplicand has to be taken away as often as is expressed by the negative multiplier; but the taking away of a negative quantity is just the same as the adding of a positive one; and therefore if there is nothing but the product of the two negative factors, that product must be exactly what it expresses greater than 0, that is, it must be the same as if both factors had the sign +, or were positive. Thus, the product of $-a$ by $-a$, or of $+a$ by $+a$, is equally $+a^2$, or without the sign, a^2; -4 by -2, is 8, in the same manner as 4 by 2; and the same in all cases.

This principle, which must be carefully attended to, is usually stated empirically in the books "like signs give +, unlike signs —;" but this, though true in the case of two factors, or of any even number, as of four, six, eight, and so on, is not true in the case of odd factors, one, three, five, seven, and so on.

We must admit *one* factor into the series of multiplications; for, as every quantity is positively *once itself*, and not any other quantity, every quantity must be considered as itself multiplied by $+1$, that is, by the integer number 1; and if we were to multiply it by -1 we should change the sign, and along with that the value of the quantity by double of whatever it expressed before being so multiplied. As this co-efficient or factor $+1$, or 1, is inseparable from the very nature of every quantity, it is never written; we do not write $1a$ for instance, because when we see a standing alone, we see at once that there is *one*, and no *more*. But if we consider it as multiplied by -1, it is quite another matter; for the difference between 1 and -1 is not only 2, but the one is 1 more than 0, and the other is 1 less than 0; and therefore multiplying by -1 changes the sign, or turns a + quantity to —, and a — quantity to +. By separating a into the factors $+1$ and a, or -1 and $-a$, and b into $+1$ and $-b$, or -1 and $+b$, it would be easy to show the truth of the rule for the signs in a manner different from the above; but we trust that which we have said will render it abundantly plain.

When a factor consists of more than one term, the multiplication of it cannot be expressed by annexing the other factor without any sign. Thus, if the one factor were $a + b$ and the other c, then neither $a + bc$, nor $ac + b$, would express the product, for the multiplication applies only to b in the one case, and only to a in the other. We must therefore mark the compound factor as a whole, by the vinculum $\overline{a + b}$, or rather by the parenthetical characters $(a + b)$; and then $(a + b)\,c$ expresses the product. If two factors are compound we must inclose each in parentheses, and then it is usual to indicate the multiplication by a dot (.), or rather by the sign $\times$, the last of which is preferable, as the dot is apt to be confounded with the full stop in common language, or the decimal point in arithmetic; thus the product of $a + b$ as one factor by $c + d$ as another factor, is expressed by $(a + b) \times (c + d)$.

As these parenthetical characters do not stand for quantities or relations, but merely point out that which is expressed in two or more parts separated by $+$ or $-$, it may not be amiss to point out, by an example in numbers, the necessity of attending to them. For this purpose let $a = 6$, $b = 4$, $c = 8$, $d = 7$, and the above expression will be $6 + 4 \times 8 + 7$ without the parentheses, and $(6 + 4) \times (8 + 7)$ with them. In the first, the multiplication extends no farther than 4 and 8, which produce 32, and there is 6 and 7, or 13 to add, making in all 45, as the whole value. In the second, the multiplication extends to the two sums 10 and 15, and their product, which is 150, is the value, which is very different from the former.

The fourth general relation of the parts of a compound quantity is that in which the one part is a dividend and the other a divisor, the value being a quotient, which we are in the mean time merely to express, not actually to find. Indeed, if the expression bears the most simple form, that is, if the divisor and dividend are each expressed by a single letter, if these letters are different, and if nothing farther is stated than that the one of them is to be divided by the other, there are not data sufficient for finding the quotient as a separate quantity. Thus, if the quotient of a divided by b is sought; and if we merely know that a stands for one quantity and b for another, but do not know what kind of quantities they are, whether of the same kind with each other or of different kinds, we cannot tell whether the quotient is or is not a quantity which we can or cannot express in any other way than by indicating it; and, even if we know that both quantities are of the same kind, so that the quotient must be a number, we are not in a condition for stating whether the quotient shall be greater than the number 1, equal to it, or less, unless we know that the quantities are equal or unequal, and in the case of inequality, which is the greater and which the less. Therefore, all that we can do in such cases is to indicate that there is a division to be performed; and this is done generally by writing the dividend above a line, and the divisor below the same. Thus $\frac{a}{b}$ indicates the quotient of a divided by b, though without pointing out what that quotient may be.

The quotients of all quantities may be indicated in the same manner: as, $\frac{a+b}{c-d}$ indicates the quotient of the upper quantity by the under, whatever may be the forms in which they are expressed. Division may also be indicated by writing the dividend, then the sign $\div$, and lastly the divisor. Thus $\frac{a}{b}$ and $a \div b$ have the same meaning, and are read "a divided by b."

We shall point out in another section how the division is to be performed so as to obtain the quotient as a separate quantity in all cases where that is possible; but there are some general principles which we can perhaps better explain in this simple view of the matter.

We may, for instance, determine the sign of the quotient, whether we can or cannot express its value by a separate quantity. Here we must bear in mind that the dividend is always equal to the product of the divisor and quotient, so that the finding of a quotient resolves itself into the finding of a quantity the product of which and the quotient shall be equal to the dividend. From this it follows, that if the divisor and dividend have the same sign, the sign of the quotient must be $+$; and if they have different signs it must be $-$; but that in the case of the same quantities as divisor and dividend, the expression for the quotient will be the same quantities whatever may be the signs.

Let us illustrate this by the simplest case that can occur, the division of a quantity by itself, or $\frac{a}{a}$. The quotient of this, in all cases of the signs, will be expressed by the number 1, because any quantity is, of course, just once itself, and nothing either over or wanting. Now, if it is $\frac{a}{a}$, or $\frac{-a}{-a}$, the quotient will be 1, that is, $+1$; but if it is $\frac{+a}{-a}$, or $\frac{-a}{+a}$, the quotient will be -1; for $-a$, the divisor in the first case, multiplied by -1, the quotient, produces $+a$, the dividend in the first case; and $+a$, the divisor in the second case, multiplied by -1, the quotient, produces $-a$, the dividend in that case. Hence the quotient of quantities which have the same sign is always a positive quantity, or as much greater than 0 as its whole value expresses; but the quotient of two quantities with different signs is always as much less than 0 as its whole value expresses. Hence, in a compound quantity, the quotient of $\frac{+a}{+b}$, or of $\frac{-a}{-b}$, may always be expressed by $+\frac{a}{b}$; and the quotient of $\frac{+a}{-b}$, or of $\frac{-a}{+b}$, by $-\frac{a}{b}$.

If the divisor were a, and the dividend $3a$, it is evident that the quotient would be 3, $+3$ if the signs were the same, and -3 if they were different; and generally, if the divisor were a, and the divident na, that is, any number of times a, whether expressible in terms of the arithmetical scale or not, the quotient would be n, that is, the same number, $+n$ if the signs were the same, and $-n$ if they were different. Now 3 is 1×3, and n is $1 \times n$, whatever number n may stand for; therefore, multiplying the dividend by any quantity has the same effect as multiplying the quotient by the same quantity.

If the dividend were a, and the divisor $3a$, the quotient would be one-third part of 1; and if the dividend were a, and the divisor na, the quotient would be the nth part of 1, or $\frac{1}{n}$; therefore, multiplying the divisor produces the same effect as dividing the quotient.

Now, if multiplying the dividend multiplies the quotient, and multiplying the divisor divides the quotient, multiplying both by the same quantity, whether that quantity be one single factor or any number of factors, will not alter the value of the quotient; or, $\frac{an}{bn} = \frac{a}{b}$, whatever n may be, whether large or small, simple or compound, provided it is the same in both cases.

This principle, which is so simple that it is nearly self-evident, is a very important one in practice; and so is the converse of it, namely, that if both divisor and dividend are divided by the same quantity the quotient is not altered. This last is, in fact, the principle upon which we proceed in the common arithmetical division of one number by another; we consider the divisor as divided by itself, and thus reduced to the number 1; and we divide the dividend also by the divisor, in order to have it expressed in terms of the divisor, considered as one.

We shall be better able to see the value of these principles afterwards, and shall discover other means of perceiving their truth; so we shall now proceed to show how the elementary operations are performed algebraically.

SECTION VI.

ELEMENTARY OPERATIONS IN ALGEBRA.

As arithmetic is merely the application of the general principles of algebra to those particular cases of quantities which can be expressed by numbers according to the scale and notation of arithmetic, it follows that the elementary operations in the one must be the same as they are in the other, namely, addition, subtraction, multiplication, and division. But if the notation of algebra, as we have attempted to explain it in the preceding section, is properly understood, these operations are far more easily performed by means of the algebraical symbols than the arithmetical ones. If we write down the given quantities with the proper signs, used in the form which indicates the operation, we have an expression for the result of that operation at once; and all that we have to do farther is to find out whether the expression thus obtained can be reduced to fewer or more simple terms. In this, every case must be considered in itself; and thus every operation in algebra is the discovery of something new, instead of the performing of the same sort of drudgery over and over, as is the case in arithmetic.

The first and most general consideration is, whether the case before us can or cannot be simplified; and as it would be vain to try the cases which cannot be simplified, the knowledge of them is the first point to which we must direct our attention. Now, the principle here is a very simple and self-evident one: if the quantities are all of different kinds, that is, all expressed by different letters, or by different letters combined with different numbers as co-efficients, we cannot shorten or simplify the expression.

Thus, in ADDITION, if the sum is that of a, b, and c, there is no simpler expression for it than $a + b + c$. Also, if it is $5a$, $3b$, and $4c$, there is no simpler expression than $5a + 3b + 4c$. But if it is $5a$, $5b$, and $5c$, we can make it 5 times the sum of a, b, and c, that is $(a + b + c) \times 5$. If the letters are the same, we can bring them into one expression; thus $5a + 3a + 4a$ is $= 12a$. Also, if the letters are the same, and some of them $+$ and others $-$, we can get one expression for the whole by taking the $+$ into one sum and the $-$ into another; subtracting the co-efficients and prefixing the sign of the greater co-efficient to the difference. Thus $5a - 5a = 0$; $5a - 4a = a$; $5a - 10a = -5a$; and so in other cases.

Also, if we have a $+$ quantity to add to any expression, and there is a $-$ quantity of the same kind, that is, expressed by the same letter or letters in the expression, we get rid of as much of the $-$ quantity as is equal to the $+$ one. Thus, if we have to add cb, that is $+cb$, to $a - cb$, the sum becomes

a; if we have to add $4b$ to $a - 6b$, the sum becomes $a - 2b$; and if we have to add $3b$ to $a - 2b$, the sum becomes $a + b$.

When we have mentioned that all the quantities of which the sum is required, express that sum when they are written one after another with their proper signs; that the only cases in which that expression can be shortened are those in which the same quantity occurs more than once; that quantities which do occur more than once in the expression may be reduced to one occurrence by taking their sum; that this sum is the sum of the co-efficients if the signs are the same, but the difference, and having the same sign with the greater, if some have $+$ and others $-$.

Any instructions more minute than these, and especially any formal or empirical rules for the adding of quantities, are not only superfluous but injurious to those who wish to understand algebra. Algebra is the art of finding out how things are to be done; and thus, if there are rules and formulæ to be learned, as a child cons by rote a catechism without understanding one word of the reason or truth of the dogma (it is in the manner not the matter that the dogma consists), that which is worked at (we will not say learned) is not algebra, it is the practice of arithmetic in an algebraical dress, more difficult, and therefore less useful than simple arithmetic, just as the common calculations of the schools are more difficult and less useful than the ready-reckoner.

In SUBTRACTION, we have only to write the quantity to be subtracted after the other quantity, connecting them by the sign $-$; and the expression thus obtained is the difference, which may or may not be shortened according to circumstances, as is the case with the sum in addition.

We must attend, however, to what is meant by prefixing the sign $-$; for, in order that we may fairly express the subtraction, this sign must affect every term which has to be subtracted. An example will perhaps show this more clearly than words.

Let it be required to subtract $5a + 3b - cd$, from $7a + 3b$; and the expression will be

$$(7a + 3b) - (5a + 3b - cd).$$

The sign $-$ affects all the three terms of the last expression within the parentheses; that is, it makes each of them -1 times itself: -1 must thus be considered as a factor or multiplier of all the terms; and we already know that the effect of -1 as a multiplier is to change the signs. Thus, one expression in the example becomes

$$7a + 3b - 5a - 3b + cd;$$

and, as this expression is perfectly general, for a, b, c, and d may stand for any or for all possible quantities, we have at the same time found this general principle: the subtraction of quantities is expressed by writing them down with their signs changed.

Let us now look back at the expression, and see whether we can shorten it. There are $+7a$ and $-5a$, which taken together make $+2a$, $+5a$ and $-5a$ being $= 0$. Again, there are $+3b$ and $-3b$, which together are equal to 0. Therefore $2a + cd$ is the difference between $7a + 3b$ and $3a + 3b - cd$.

Let us try by addition if the quantity subtracted and the difference make the other quantity, that is, if

$$5a + 3b - cd + 2a + cd = 7a + 3b.$$

Looking at the quantity to the left of the sign $=$ we find $3b$, and there is $3b$ in that to the right; thus one term in each agrees. Again, we have $5a + 2a$ on the left, and $7a$ on the right, all with the sign $+$, therefore there is in effect $7a$ on both sides, and these again agree. Farther, we have $-cd$ on the left, and no $c\,d$ with any sign on the right, but when we look farther at the left side of the sign we find $+ c\,d$ as well as $- c\,d$; and $+ cd - cd = 0$. Thus we have in effect $7a + 3b = 7a + 3b$, which are not only equal but the same identical quantity.

We may mention that we can never use the sign $=$ unless the quantities to the left of it, taken altogether or as one whole, are exactly equal to those on the right, taken as one whole; and that when we can bring them to an identity of expression without changing their values in *respect* of each other, we prove this equality.

An expression of this kind is called an EQUATION; and it is the general mode of proceeding in algebra, whether the object be the establishment of a truth, the investigation of a principle, or the finding of an unknown quantity. Indeed, it is the universal formula in the acquiring of all knowledge, of whatever kind it may be; for it is by a perception of the equality either in things themselves or their relations, and by that alone, that we can pass from the known to the unknown. We are on the known bank of the river, and the unknown is the other bank, relation, the foundation and standard of which is equality, is the means by which we are to pass the river. We have the boat in some cases and only "inflated bladders" in others; but in algebra we have the bridge always open, and "no pontage" *after we know it.*

Even in the most common case of arithmetic, that of the addition of two or more simple numbers, there is an equation involved; and if we wish to understand even that simple case well, it would be better to state this equation at the beginning. The stating of the equation is, as we shall be better able to explain afterwards, nothing more than noting down what we have to do before we begin the doing of it; and everybody knows that this is not only the best way to ensure success, but the way of finding out whether what we intend to do be or be not possible.

Let the arithmetical problem be to find the sum of 2, 3, and 5; call the sum s, then we have this equation

$s = 2 + 3 + 5$; and shortening, we have $s = 10$.

In an equation, the quantities which stand to the right and left of the sign $=$ are called the sides of the equation: if they are identical, that is, if they are exactly the same expressions, the equation is of no use; for we need no algebra to tell us that anything is equal to itself. But if the two sides are differently expressed at first, and we can bring them to an identity of expression, keeping the equality all the time, we thereby prove the equality of the first statement.

Hence, an equation would be of no use to us if we could not alter the expression of it without altering the equality of the sides. Now it is almost a truism to say that equal quantities are equal, equal additions are equal, equal subtractions are equal, equal multiplications are equal, and equal divisions are equal; and therefore we may generalise them all into this one statement, the truth of which nobody who understands the words can doubt: if two quantities are originally equal, and if whatever is done to the one of them be also done to the other, they must remain equal during any succession of changes, however many, and be equal at the end. Simple and self-evident as this seems, it is the general principle of algebra, and it is quite sufficient for the purpose. It also has the advantage of being understood the instant it is stated, and remembered as soon as it is understood. We have introduced it thus early, because it is of importance that we should take it with us from the beginning, as a means of satisfying ourselves of the truth of even the most elementary operations.

To return to subtraction:—as subtracting is performed by merely changing the signs of the quantities subtracted, from + to —, or from — to +, it follows that $a-b$ is the remainder after $+b$ is taken from $+a$, and that $a+b$ is the remainder after $-b$ is subtracted from $+a$, or, which is the same thing, it is the remainder when $-b$ is subtracted from a. In all cases, the expression $a+b$ is a positive quantity, or greater than 0; but $a-b$ is greater than 0 only when a is greater than b, or by the sign, when $a \urcorner b$; and when $a \angle b$, then $a-b$ is negative, or less than 0.

a and b may be any two quantities whatever, provided that they are so far of the same kind as that they can be said to be equal, or the one greater than the other. Hence $a+b$ is the sum, and $a-b$ the difference of any two quantities whatever. Let us add them together, and see if we can learn anything from their sum. To add them we have only to connect them by the sign +, which, as we have already seen, does not alter the signs; so we have, without any shortening, $a+b+a-b$. In order to shorten this, we find that we have a twice over with the sign +, which may be expressed by $2a$; and we have $+b$ and $-b$, which together are $= 0$. Therefore, the sum of $a+b$ and $a-b$ is $= 2a$; but $a+b$ is the sum of any two quantities, and $a-b$ is the difference of the same. Therefore, in words, the sum and difference of any two quantities are together equal to twice the greater quantity; and consequently, half the sum and half the difference are together equal to the greater quantity. But if a were the less quantity, $a-b$ would be a *minus* quantity, and the sum of $a+b$ and $a-b$ would be that *minus* quantity less than $a+b$; but that quantity would be the difference, the same as in the other case; and thus we have the difference between the sum and difference of any two quantities equal to the less quantity, or, the less quantity is equal to half the sum — half the difference.

We may state the problem formally thus. Give the sum and difference of two quantities to find the quantities themselves; and the answer according to the above investigation is, to half the sum add half the difference for the greater, and from half the sum take half the difference for the less.

Simple as it must seem when thus investigated, this is both a general and an important principle. Let us try it by an example: a man has 40 shillings in both pockets, and 6 more in the right than in the left, how many are in each? 40 is the sum, 6 the difference, 20 the half sum, 3 the half difference; therefore $20+3$, or 23 in the right pocket, and $20-3$, or 17 in the left pocket, and $23+17=40$, and $23-17=6$, which answers the conditions.

There is another point connected with subtraction which is worth knowing and keeping in mind, and that is, what operations, performed equally to each of two unequal quantities, alter, or do not alter, the difference of those quantities. In compound quantities, this often enables us to see the difference at once, and without any labour.

In order to understand this, let us suppose a greater than b, and that the difference is d, which will include all quantities whatever, and consequently all differences; the greater is equal to the less and the difference, or $a=b+d$; consequently $a > b$, or $b+d > b$, and d is the difference; for, take away d, and we have $b=b$, which is self-evident. Now if we add to b, in both cases, any quantity whatever, simple or compound, which for shortness we may call q, we have $b+q+d > b+q$, and the difference d, as before; so also, if we subtract any quantity, q, we have $b-q+d > b-q$, and the difference d, as before; for, if we take from the equal quantities on both sides of the sign $>$ in each,

b in the first instance, $b+q$ in the second, and $b-q$ in the third, we have in each $0+d \urcorner 0$, which is saying, in other words, that all that remains is $d \urcorner 0$, or simply d. Hence we may state generally that, if we add equally to, or subtract equally from each of two unequal quantities, we do not alter their difference; or, that the difference of two quantities must remain the same, if we do not add or subtract a difference.

But, on the other hand, if we apply a difference to the two quantities, either by adding more to the one of them than to the other, or by subtracting more from the one of them than from the other, the original difference must be altered to the full amount of the difference so applied. Thus, if $a=b+d$, and consequently $b+d \urcorner b$, and $e=f+g$, and consequently $f+g \urcorner f$.

First, if we add the greater to the greater, and the less to the less, we have $b+d+f+g \urcorner b+f$; and, taking away the equals, we have $d+g \urcorner 0$, or the sum of the differences.

Again, if we subtract the greater from the greater, and the less from the less, we have $b+d-f-g \urcorner b-f$; and taking away the equals, $b-f$, we have $d-g \urcorner 0$, that is, the difference of the differences; and this will be greater than 0, equal to it, or less, according as d is greater than g, equal to it, or less.

Thirdly, if we add the less to the greater, and the greater to the less, we have $b+d+f$ and $b+f+g$; and, leaving out the equals, we have d and f, the difference of which is $d-f$, and may be greater than 0, equal to it, or less, as in the former case. The expression for this uncertainty is, $d \urcorner = \angle g$. This is the general expression for pointing out that two quantities are of the same kind; it comprehends every variety of value in the quantities, and therefore the principle which it embodies is of great use in the doctrine of proportion.

If we were to name the value of $d-g$, or, which is the same thing, of $a-b$, by another letter, the sign of that letter would be ambiguous: if a were the greater, it would be $+$; but if b were the greater, it would be $-$; but as they are perfectly general, either of them may be the greater, and there are many cases in which we are not able to say which. The general expression for all cases in which it is not known which is the greater, is, $a-b=\pm d$. If a is the greater, it is $+d$, or d; if they are equal, it is $0\,d$, or 0; and if a is the less, it is $-d$. This double sign, $\pm$, which is read "*plus* or *minus*," is the most general expression for a difference, when that difference is expressed by a quantity: thus, $a\pm d=b$ is an equation, whether a be greater than b, equal to it, or less.

When the comparison is made by the triple sign, $7=\angle$, which is read "greater, equal, or less," it is the *whole* of the one quantity which is compared with the *whole* of the other, and the middle point is that of perfect equality between the two as wholes. But when the comparison is made by the double sign, $\pm$, which is read "*plus* or *minus*," that is, "more or less," it is the difference to which the attention is directed, and the middle point is 0, which happens when the quantities are equal. The double sign $\pm$, therefore, properly belongs to addition and subtraction; the triple sign, $7=\angle$, does not.

From what has been said, it will be seen that, generally, the sum of all the differences of any number of quantities is equal to the difference of their sums, although some of the particular differences be $+$, and others $-$. An example will illustrate this:—

A man has two pockets on each side of his coat, one on each side of his vest, and one on each side of his trowsers. His pockets are loaded with penny-pieces; 47 right, 39 left, in the trowsers; 31 right, 35 left, in the vest; 87 right, 95 left, in the skirt-pockets of the coat; and 63 right, and 45 left, in the breast-pockets: how many must he bring from the one side to the other, to be equally loaded right and left?

Take the right side; then the trowsers give $+8$, that is, 8, the vest -4, the skirts -8, and the breast-pockets $+4$, or 4; but $8-8+4-4=0$; therefore he is already equally loaded right and left.

It only remains to be considered how the difference of two quantities is affected by the multiplication and by the division of the quantities themselves, and this can be better understood after we have examined the general principles of multiplying and dividing a little more intimately than can be done by common arithmetic, where the disappearance of the original numbers prevents us from seeing how they are combined, so as to produce the result. We shall, therefore, proceed to the third general operation.

MULTIPLICATION of quantities. In the case of simple quantities, or those expressed each by one letter only, there is no operation to be performed, and we have merely to express the product by the sign $\times$, or by writing the letters after each other without any sign; thus, $a \times b$, or $a\,b$, is the product of the two factors, a and b, and there are no means by which we can express that product more simply. When the same factor occurs twice or oftener, we may use an exponent to show the number of times; thus, $a\,a\,a\,a$ may be expressed a^4, and $aaabb$ may be expressed $a^3\,b^2$, it being always understood that every letter in such an expression occurs as often, as a factor, as is expressed by the exponent.

As two factors are required for the first multiplication, there is always, in the case of a single letter, one actual multiplication fewer than the exponent indicates; thus, a^4 is a multiplied three times by a. But when there are other factors, the product of these is understood to be multiplied by the letter which has the exponent, as often as is expressed by that exponent; thus, $a^3 b^2$ means a^3 multiplied twice by b, or b^2 multiplied thrice by a.

This impossibility of shortening the product of any number of simple factors, all different from each other, though a very simple matter, is not an unimportant one, for it shows us that we must make actual observation the foundation of all our operations, and that all the facilities which science can give us are only judicious methods of managing these results of observation, which are equally open to everybody. The result of a multiplication is always a number of times a number, whatever value we may attach to the unit, or individual 1, of one or of both of the numbers; and when we consider each number as one whole, as we do in the case of all numbers not greater than 9, we have no way of finding the product but by actually counting. There is, however, a way in which we can derive those products from each other; and this is the way in which we express the products of compound quantities generally or algebraically.

It will be borne in mind, that a compound quantity, according to our notation, is a quantity consisting of two terms, or more, separated from each other by + or —. $a+b$, or $a-b$, is the simplest expression which we have for a quantity of this kind, and it may be considered as perfectly general, for the letters a and b may be considered as representing any quantities whatever, whether simple or compound; and therefore it will follow, that whatever can be shown to be true of their multiplication, will be true of the multiplication of all possible quantities. $a + b$, or $a - b$, taken as a whole quantity, is called a *binomial*, from the fact of there being two named or at least nameable parts in it, of which it is the sum when the sign is +, and the difference, with the sign + or —, according to circumstances, when the sign is —. Hence, if we fully understand the composition of the product of $a+b$, and $a-b$, each taken both as multiplier and multiplicand, we shall be in possession of the general principle which applies to the multiplication of all compound quantities; and the products of these quantities will involve the general principle of the multiplication of all compound quantities whatsoever.

Now a compound quantity may be considered as made up of, and exactly equal to as many parts as there are terms in it; and we evidently express and mean the very same value, whether we state the entire compound as one whole, or all the parts of it. Thus, in the case of the number 7, it is the same in amount whether we write 7, or $6+1$, or $5+2$, or $4+2+1$, or any number of parts which, taken altogether, amount exactly to 7; and generally, if $c=a+b$, either expression may be used instead of the other; and if we multiply the parts $a+b$ by any factor whatever, it must produce the very same result as if we multiplied the single quantity c, which is $=a+b$, by the same. Hence, to multiply any compound quantity, we have only to multiply all its terms.

Also, if we have to multiply by a compound quantity, we may multiply by all its terms or parts; for $a+b$ times any quantity whatever, is evidently the same as a times that quantity $+\ b$ times; and $a-b$ times any quantity is evidently a times that quantity $-\ b$ times.

A multiplier with the sign $+$ will, therefore, produce no change on the signs of the quantity multiplied; but a multiplier with the sign $-$, as it gives a product to be subtracted, will change all the signs.

It has been already mentioned that the products of compound quantities may be indicated by inclosing each quantity in parentheses, and connecting them by the sign $\times$; thus, $(a+b)\times(a+b)$, is the product of $a+b$ by $a+b$; but as the whole multiplier $(a+b)$, affects equally each of the terms of the multiplicand, the terms of the above expression are not simple.

We might simplify them one step by connecting the multiplier with each term of the multiplicand; and as, by the general principle of algebraic notation, multiplication is always understood when there is no sign between letters, we should have the product $a(a+b)+b(a+b)$, in which a is a multiplier of both a and b in the first compound term, and b a multiplier of both a and b in the second.

Next we might actually apply these multipliers, which is done by simply placing the multiplying letter along with each of the others, either before them or after them, without any sign between; and then we should have $aa+ab+ab+bb$, for the product of $a+b$ by $a+b$.

But this product can be shortened, for aa is a^2, $ab+ab$ is $2ab$, and bb is b^2. Thus the product of $a+b$ by $a+b$, is $a^2+2ab+b^2$.

$a+b$ is the sum of any two quantities whatever; and the product of any quantity, simple or compound, by itself is called the *square* of that quantity; therefore the square of $a+b$, or, as we may express it, $(a+b)^2$, is $a^2+2ab+b^2$.

This, though a very simple result, is a very important one, as we shall be better able to see afterwards; but it will be useful in the meantime to look at the composition of this square of $a+b$: it consists of three terms, $a^2+2ab+b^2$, the first of which, a^2, is the square of a; the last, b^2, is the square of b; and the middle one, $2ab$, is twice the product of a and b. But a and b are any quantities whatever, and therefore whatever is true of the square of them, must be true of that of any quantity or number whatever which we can imagine to be the sum of two parts. Wherefore, we have this general conclusion from the above simple operation:—the square of the sum of any two quantities is equal to the sum of their squares, and twice their product. This is a very important principle, and one to which we shall have frequent occasion to revert, therefore the student of mathematics must bear it carefully in mind.

Let us now consider the case in which the simple quantities of which the multiplicand and multiplier are made up, are all different from each other; as, for instance, the product of $a+b$, multiplied by $c+d$: we may arrange them in the common way of arithmetical multiplication, as follows:—

$$\begin{array}{lll} & a+b & \\ & \underline{c+d} & \\ & ac+cb & \\ & \quad\; \underline{ad+bd} & \\ \text{Product,} & ac+cb+ad+bd. & \end{array}$$

No two of the terms in this result are similar, and consequently it cannot be shortened; therefore the only conclusion that we can draw from it is, that the product of the sum of two quantities by that of other two, both different, is equal to the sum of the four products which arise from multiplying the first by the first of each, the second by the second, and the first of each by the second of the other. But even this is something, for, let it be required to find the product of 17 and 15: we may call 17, $10+7$, and 15, $10+5$; then we have the product, $10\times10+10\times7+10\times5+7\times5$. Performing the multiplications, we have, $100+70+50+35$; and adding these we have 255. This is not shorter than the common method of multiplying, but it shows us how the product is composed, for we may obtain the product of the sum of any two numbers by the sum of any other two, by taking the products of the first of each pair, the second of each pair, and the first of each pair by the second of the other, and adding them together.

Let us now examine the product by itself, or, which is the same thing, the square, of a binomial having the sign — in the second term; that is, the product of $(a-b) \times (a-b)$, or $(a-b)^2$.

Multiplying by a, we have a^2-ab; and multiplying by $-b$, and changing the signs, because b is —, we have $-ab+b^2$; or we might have arranged them thus:—

Multiplicand	$= a-b$
Multiplier	$= a-b$
Product by a	$= a^2-ab$
Product by $-b$	$= \quad -ab+b^2$
Product by $a-b$	$= a^2-2ab+b^2$, or, a^2+b^2-2ab.

$a-b$ is a general expression for the difference of any two quantities; it is a positive quantity, or greater than 0, when a is the greater, equal to 0 when a and b are equal, and a — quantity, or less than 0, when b is the greater. It will be borne in mind, that any difference may be considered either as a + or a — quantity, for the greater is equal to the less, + the difference, and the less is equal to the greater, — the difference.

The above expression for the square of the difference contains a^2+b^2-2ab, that is, the sum of the squares wanting twice the product. From this we see that, if there is any difference between two quantities, the sum of the squares of those quantities must always be greater than twice their product; and that this difference being = to the square of the difference of the two quantities, does not depend on the value of the quantities themselves, but applies to all which have the same difference, whether they be great or small. If $a=b$, or the difference = 0, then the square of the difference = 0, and consequently $a^2+b^2=2ab$, which is only stating, in other words, that the squares of equal quantities are equal, and that the product of two equal quantities is equal to the square of any one of them.

This shows us, that when the factors are equal, the product is the greatest possible, and this is sometimes useful.

Let us now examine the product of the sum $a+b$, and the difference, $a-b$, or $(a+b)\times(a-b)$. We have,

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\; -ab-b^2 \\ \hline a^2-b^2 \end{array}$$

Here $-b$ changes the signs in the second line of the product, and $+ab-ab$ is $=0$, and may be left out in adding. Hence we have this principle:—the product of the sum and difference of two quantities is equal to the difference of their squares.

If we subtract the square of $a-b$ from that of $a+b$, we have,

$$\begin{array}{rl} (a+b)^2 = & a^2+2ab+b^2 \\ (a-b)^2 = & a^2-2ab+b^2 \\ \hline & 4ab = \text{difference.} \end{array}$$

Therefore, four times the product of any two numbers, together with the square of their difference, is equal to the square of their sum.

We may now shortly examine what changes are made upon the product, by additions to, or subtractions from, either or both of the factors. For this purpose, let a and b be any two factors whatever, of which the product is ab; add to the factor

a, any quantity c, and the product becomes $ab+bc$; subtract any quantity c, and it becomes $ab-bc$; let the change be made on the other factor b, by any quantity d, while a remains the same, and we have $ab+ad$ when d is $+$, and $ab-ad$ when d is $-$. Let each have any quantity added, as $(a+c)\times(b+d)$, and we have $ab+bc+ad+dc$; and let any quantity be subtracted from each, and we have $(a-c)\times(b-d)$, from which we obtain $ab-bc-ad+cd$.

These expressions give us the following general principles:—

1st. If we add to either factor, we add to the product the same number of times the other factor; or, the product of the sum is equal to the sum of the products.

2nd. If we subtract from either factor, we subtract from the product the same number of times the other factor; or, the difference of the products is equal to the product of the difference.

3rd. If we add to each factor, we add to the product the product of each factor by the quantity added to the other, and also the product of the two quantities added.

4th. If we subtract from each factor, we subtract from the product the product of each factor by the quantity subtracted from the other, and add the product of the two quantities subtracted.

5th. If we add to the one factor, and subtract from the other, we have $ab+bc-ad-cd$; we must add the product of the one factor by the quantity added to the other, and subtract the product of the $-$ quantity by the other factor, and also the product of the quantity added and the quantity subtracted.

These principles often enable us to get at products by means of sums and differences. Thus, 100×100 is, by the scale of numbers, 10000: then let us find the product of 109 and 107.

This falls under the third case $(a+c) \times (b+d)$, or $(100+9) \times (100+7)$; so we have to add the following lines:—

$$\begin{aligned} ab &= 100 \times 100 = 10000 \\ bc &= 100 \times 9 = 900 \\ ad &= 100 \times 7 = 700 \\ cd &= 9 \times 7 = 63 \end{aligned}$$

$$\text{Therefore } 109 \times 107 = 10663$$

Again, to find the product of 97 and 93. This falls under the fourth case, $ab - ad + bc - cd$; and

$$\begin{aligned} ab &= 100 \times 100 = 10000 \\ -bc &= -100 \times 3 = -300 \\ -ad &= -100 \times 7 = -700 \\ cd &= 3 \times 7 = 21 \end{aligned}$$

$$\text{Therefore } 97 \times 93 = 9021$$

Farther, let the product of 109 and 94 be required. This comes under the fifth case, $ab + bc - ad - cd$; and

$$\begin{aligned} ab &= 100 \times 100 = 10000 \\ bc &= 100 \times 9 = 900 \\ -ad &= -100 \times 6 = -600 \\ -cd &= -9 \times 6 = -54 \end{aligned}$$

$$\text{Therefore } 109 \times 94 = 10246$$

These principles are often of great use to us in shortening operations in common arithmetic; thus, a table is 6 inches less than 9 feet long, and four inches more than 6 feet wide, how many square feet are in it? Here we have $a = 6$ feet, $b = 9$ feet, $c =$ one-third of a foot ($\frac{1}{3}$), and $d =$ one-half of a foot ($\frac{1}{2}$); so that in numbers we have $(6+\frac{1}{3}) \times (9-\frac{1}{2})$; and

$$\begin{aligned} a \times b &= 6 \times 9 &&= 54 \text{ feet.} \\ b \times c &= 9 \times \tfrac{1}{3}, \text{ or one-third of } 9 &&= 3 \\ -a \times d &= -6 \times \tfrac{1}{2}, \text{ or one-half of } 6 &&= -3 \\ -c \times d &= -\tfrac{1}{2} \times \tfrac{1}{3}, \text{ or one-sixth of } 1 &&= -\tfrac{1}{6} \end{aligned}$$

$$\text{Therefore } (6 + \tfrac{1}{3}) \times (9 - \tfrac{1}{2}) = 53\tfrac{5}{6} \text{ feet.}$$

This example is rather an anticipation of principles which have not yet been explained; but it is simple, and may help to show that these methods of considering the composition of a product are very useful.

As the product of any three factors is the same as the product of any two of them by the third one, a, b, c being either $ab \times c$, or $ac \times b$, or $bc \times a$, it follows that the multiplication of any one factor, before multiplying them together, is the same as multiplying their product after; and if both factors are multiplied each by any quantity, the product will be the same as if the product of the original factors were multiplied by the product of the multipliers. Thus, if the factors are made $n\,a$ and $m\,b$ before multiplication, the product of $n\,a \times m\,b$ is the same as that of $a\,b \times m\,n$. Hence we see generally that, if a product is made up by the continual multiplication of the same factors, it is of no consequence in what order they are taken.

Products which are the result of multiplications by the same, or by equal multipliers, are called *equi-multiples*; thus, $3\,a$, $3\,b$, and $3\,c$ are all equi-multiples of the quantities a, b, and c, by the common multiplier 3; and $m\,a$, $m\,b$, and $m\,c$ are equi-multiples of a, b, and c, by the common multiplier m, which may stand for any number or quantity whatever. These principles, though simple, are very important.

Some of the conclusions at which we have arrived, on mere inspection of the results of algebraical operations, will be found of great use when we come to apply them, and in the meantime they will show the great clearness and simplicity of this science, and that we have only to learn how to read and to write algebraical expressions, in order at once to see the truth of what they contain. To know this is of far more importance than the mere performing of multiplication, or any other process, by technical rules; and though there are many results which it is necessary to remember, and which, indeed, it is difficult to help remembering, yet especial care should be taken that not one algebraical fact is committed to memory without being previously traced from self-evident principles, and by that means thoroughly understood.

We shall now point out how terms which are in themselves a little more complicated than those which we have hitherto mentioned are treated in multiplication. The rule of the signs is necessarily the same in all cases; that is, a + multiplier does not change the signs of the multiplicand, but a — multiplier always does.

Besides this, we have only to consider the parts of which a term may be composed. These are: first, numbers, or numeral co-efficients, which are always placed first, and are understood to multiply the whole term; that is, their influence extends as far as the next + or — in a compound quantity. Secondly, general quantities expressed by letters; and each of these is also understood to multiply the whole term, or as far as the next + or — in a compound quantity. Thirdly, exponents, which apply only to the single letter or number over the right of which they are written, unless more than one are inclosed in parentheses, and the exponent placed immediately after.

Thus, in the expression $4abc^2$, the meaning is, that the number 4, the quantity a, the quantity b, and the square of the quantity c, are all multiplied together. If the expression were $4a^2b^2c^2$, it would mean that the squares of all the quantities expressed by letters were to be multiplied together, and also by the number 4. If it were $4a(bc)^2$, it would mean that 4, the quantity a, and the square of the product of b and c, were all to be multiplied together. It is needless to repeat that, when there is no numeral co-efficient, 1 is always understood, and might be written, for every expression, whatever it may be, is of course exactly once itself.

Numbers must be dealt with arithmetically, that is, they must be actually multiplied, or else the multiplication of them must be indicated by the sign $\times$ between them.

Letters must be dealt with algebraically, that is, they must be written after each other, without any intervening sign; or, if the same letter occur more than once, it may be put only once, with an exponent expressing the number of times it occurs.

Exponents, when they are attached to the same quantities, that is, the same letters, or combinations of letters, must be dealt with as exponents, that is, they must be added, and their sums will represent their products.

The following example, in which all these kinds of quantities occur in one or other of the terms, will serve to illustrate the application of the principles to every case of multiplication.

Let it be required to multiply $3a^2b+3ab-3b^2$, by $4ab^2-6ab+2b^2$. Arrange the quantities as in arithmetic, multiply by each term in its order, add, and shorten the sum, if possible.

$$
\begin{array}{l}
3a^2b + 3ab - 3b^2 \\
4ab^2 - 6ab + 2b^2 \\
\hline
12a^3b^2 + 12a^2b^3 - 12ab^4 \\
\qquad\quad -18a^3b^2 - 18a^2b^2 + 18ab^3 \\
\qquad\qquad\qquad\qquad 6a^2b^3 + 6ab^3 + 6b^4 \\
\hline
12a^3b^3 - 18a^3b^2 + 18a^2b^3 - 18a^2b^2 + 24ab^3 - 12ab^4 - 6b^4
\end{array}
$$

It is evident that the number of terms in the whole product of two compound quantities, must be equal to the product of the numbers in both; that it never can exceed this, and that it cannot be less, except two or more of the particular products consist of the same letters, with the same exponents; and then such quantities can be united into single terms, by the same principles which were explained in Addition.

Thus, in the above example there are nine terms, arising from the multiplication of each of the three terms of the multiplicand by each of the three terms in the multiplier; but when we examine them, we find $12a^2b^3$ as the second term of the first line, and $6a^2b^3$ as the first term of the last line, and both with the sign $+$; so they together make $18a^2b^3$. Also we find $18ab^3$ as the last term of the second line, and $6ab^3$ as the second term of the last line, both with the sign $+$; thus they make $24ab^3$ in the general product: by this means our nine terms are shortened to seven.

It is of great consequence to be expert in the multiplying of quantities, and also in discovering when terms are or are not of the same kind. But this, like all practical operations, can be done readily only by actual practice; and that practice is most advantageously done after the student is able to perform Division as well as Multiplication, because the one operation serves not only to prove the correctness of the other, but also very often throws considerable light upon the means by which that other is performed; and, in practising every kind of mathematical exercise, even for the mere purpose of acquiring facility in the simple operations, the student should always convince himself of the truth of every result, by seeing the connection between it and self-evident principles. We shall, therefore, proceed to give some short account of the fourth general operation in the science of quantity—Division of quantities.

It is of the utmost importance to have clear notions of what is meant by the DIVISION OF QUANTITIES, because, whether generally in Algebra, or particularly in Arithmetic, the word has a meaning somewhat more extended than in common language. The common notion of division is the separating of anything into parts, without any restriction as to number, or to the parts being equal to each other; but arithmetical division has always reference to the *value of one* of a proposed number of equal parts in a quantity, or to *the number* of parts of a given magnitude or value, into which a given quantity can be divided.

It is, in this sense, exactly the reverse of multiplication, and the actual performing of it is a mere reversal of the operation of multiplying. In real quantities, one of the two factors in multiplication must always be considered as a number, and then the product is of the same kind with the other one. Thus, if the data for multiplication be the quantity and price per unit of an article, and the product sought the price of the whole quantity at the same rate, this product, if we call the price of the unit p, and the quantity, that is, the number of that unit in the quantity q, then the whole price will be expressed by $p \times q$, or $p\,q$. It is evident that, if we divide this product, $p\,q$, by either of the two factors, the result will be the other one— p, if we divide by q, and q, if we divide by p.

But it is also evident that the product, $p\,q$, that is, the whole price of the quantity of goods, may be presented to us, not as the result of a multiplication, but as one simple and original whole, which may be represented generally by any letter, as a. If nothing is given us but this simple quantity, it is evident that we can make nothing of it; for, let $a = 100l.$, and let it be asked how much goods, or what priced goods we can buy for it; and no direct answer could be given. If, however, either p, the price per unit, or $q+$, the quantity, were given us, as well as a, we should be in a condition for answering the question. If q were given then p equal to a divided by q; and if p were given then q equal to a divided by p, which are expressed algebraically either by $a \div q = p$ and $a \div p = q$, or by $\frac{a}{q} = p$, and $\frac{a}{p} = q$, the last of which is the preferable expression, because it is least liable to be mistaken.

In this way we may indicate the division of any quantity by any other quantity, whether those quantities be simple or compound; and if the quantities are both simple, and consist of different letters, this is all the division which we can perform. This simplest case may be considered as comprising the division of all numbers which are less than ten times the arithmetical figures by those figures: this brings us to the reversal of the table of multiplication, and is the elementary operation beyond which division cannot be simplified as a practical operation.

We are, therefore, to bear in mind that $\frac{a}{b}$ is a general expression for the quotient or result of the division of any one quantity by any other, a being in all cases the *dividend*, or number to be divided, and b the *divisor*, or number by which we are to divide. Also, if we were to suppose the division performed and the quotient obtained in any number, q for instance, we would have $bq = a$, that is, the product of the divisor and quotient equal to the dividend; and if we actually performed this multiplication in real numbers, we should from it discover whether our division were right.

The dividend a may be considered as the product of the divisor b and the quotient q, whether the quotient be or be not found; and thus it becomes a very important consideration in division what changes we can perform on the dividend a, and the divisor b so as not thereby to affect the value of the quotient.

Now it is evident that whatever may be the value of the quotient q, whether greater than 1, equal to 1, or less; and whether it be possible or not possible to express it as a separate quantity, it will not be altered if we take any equi-multiples whatever of the dividend and divisor; but that $\frac{ma}{mb}$ expresses exactly the same quotient as $\frac{a}{b}$, whatever may be the values of a and b, and also of the multiplier m, provided this multiplier is the same in the case of each of them; for a is the product of b the divisor by q the quotient; and it was shown, when treating of multiplication, that multiplying one of the factors produces the same effect as multiplying the product by the same multiplier. But ma is the product of the divisor and quotient by any quantity m, and therefore the quotient of ma must be exactly m times that of a not multiplied m. Hence we may conclude generally that the multiplying of the dividend by any multiplier whatever must always have the same effect as multiplying the quotient by the same multiplier.

Let us next consider what will be the effect if we apply the same multiplier to both divisor and dividend. We have already seen that if the quotient expressed by $\frac{a}{b}$ be the quantity q, the quotient of $\frac{ma}{b}$ must be mq. Then let us consider what will be the quotient of $\frac{ma}{mb}$. Here m and b are both divisors, and they are divisors in the relation of factors, therefore, whatever number of times any dividend is divided by and divisor b, it must be as many times oftener divided by any multiplier m, which is applied to the divisor b. Consequently the quotient of $\frac{ma}{mb}$ is exactly equal to that of $\frac{a}{b}$; and we have these general principles: that, multiplying the dividend multiplies the quotient; multiplying the divisor divides the quotient; multiplying both by the same factor, or the same number of factors, of which the continual product is the same in both cases, taken in any order, does not alter the quotient; and that, if the divisor and dividend are unequally multiplied the quotient will be multiplied by the quotient of the two multipliers, taken as applied to the original terms. In the last case the change of the quotient may be either an increase or a diminution, or it may become equal to the number 1, according to circumstances, which of course depend upon the nature of the particular case.

As multiplying the one term in division has the same effect as dividing the other term by the same or an equal quantity, it follows that both may be divided by the same quantity without affecting the value of the quotient; and from this again it follows, that if we divide both divisor and dividend by any quantity, or any series of quantities of which the continual product is the same in both cases, we leave the quotient unaltered; but if we apply one divisor to the dividend, and a different one to the divisor, we change the expression to the value of the quotient of the two quotients.

Thus, we may be said to have an almost unlimited power over the expression of a quotient when it is given us in terms of a divisor and a dividend; and this is especially worthy of our consideration, as being the part of the science in which we can most easily obtain clear notions of the relations of quantities to each other in respect of value or magnitude, and as these relations are the only means which we have in finding out unknown by the known, it is necessary that we should make ourselves very intimately acquainted with their nature. We may add, that our failure is more frequently owing to ignorance on the subject of relations than to any other cause, and from ignorance too which is often occasioned by an inconsiderate supposition that this part of the subject is too self-evident and simple for requiring any thought at all.

If we suppose a and b the dividend and divisor, and q the quotient, we have the value of this quotient expressed by $\frac{q\,b}{b}$; and as dividing both terms by the same quantity does not alter the value, if we divide both of these by b we have $\frac{q}{1}$ for the value of q, which is an expression in which the value of the divisor is reduced to the number 1. But a, b, and q are any divisor, dividend, and quotient whatever, because none of them expresses any particular value; therefore we have this general principle, that every case of division, be it what it may, may be reduced to another expression for exactly the same quotient or value in which the divisor shall be the number 1.

The performing of division arithmetically is nothing more than finding the value of the dividend which shall correspond to divisor 1; and the performing of division algebraically is the finding of the fewest and simplest terms in which this dividend answering to divisor 1 can be expressed, this expression being the algebraical simplification of the quotient.

If the expression is, given $\frac{a}{b}$ to find q the quotient, and if we suppose a and b to be simple quantities, we must first consider whether they are of such a nature as that they can have a quotient; and secondly, we must endeavour to find out the value of this quotient in known terms. The first of these questions must be determined from the nature of the quantities: if they are real quantities of the same kind the quotient is a number; and if of different kinds we must be able to tell, from their nature, whether they are so related as that we can call the divisor a number, and so have for the quotient a quantity of the same kind with the dividend. If the divisor and dividend do not come within these conditions there can be no quotient to which meaning can be attached, and consequently no practical usefulness in the particular case; but when we speak of quantity generally as expressible by numbers, we may without impropriety consider all such general numbers as expressing quantities of the same kind, because none of them expresses any specific kind of quantity.

After having determined that there can be a possible quotient in the case, we are next to determine its amount; and here we are thrown upon the particular case, and can find it only by trials. We have already shown, in the section on the operations of arithmetic, that the quotient of two numbers can in all cases be obtained, either wholly or to as great a degree of accuracy as may be necessary for even the nicest purposes, by as many separate multiplications and subtractions as there are figures in the quotient; but this is not a principle which will apply generally to the finding of quotients algebraically, because the properties of the scale of numbers are peculiar, and confined to arithmetic. Hence it is impossible to lay down any general rule for the actual division of quantities which are expressed algebraically; neither in the case even of simple quantities can we tell before-hand of how many terms the quotient may consist; for we have seen that in multiplication terms often destroy each other, and thus the product becomes apparently as simple as either of the factors, or even simpler. When the terms are single letters the expression cannot be shortened; neither can it be shortened when the terms are compound quantities of which the single terms are all different; but if the same letter occurs in each term it may be entirely left out, as in that case the terms are equi-multiples by that letter as common factor, and dividing by this common factor does not affect the quotient.

To take a few of the simplest cases: $\frac{a}{b}$ cannot be shortened; $\frac{ab^2c}{ab^2}$ is evidently $= c$, because both terms are divisible by ab^2, and c is the quotient which answers to 1 in the divisor.

When there are compound quantities and terms of quantities, we are obliged to find the terms of the quotient by trial and error; and for this purpose it is desirable to take as the first term of the quotient some quantity which multiplied into the first term of the divisor shall produce at least one term of the dividend, because then the subtracting will make the remainder that term shorter. It is to be understood, however, that the length or shortness of a compound quantity has no necessary connection with its real value.

The multiplications are of course to be performed according to the directions already given; and the signs of all the terms of the products are to be changed, because these are to be subtracted. If only the same letters occur in both divisor and dividend, the division can, generally speaking, be performed; and as this is a merely mechanical part of the business, the best way in acquiring expertness in it is to practise multiplying one compound quantity by another, and then dividing the product by either factor; and if the quotient turns out to be the other factor, both operations will of course be right. The following examples will show how this is to be done.

First, if the sum of the squares of two quantities, wanting the product, is multiplied by the sum of the quantities, what will be the amount?

Let a be one of the quantities and b the other, then,

$$\begin{array}{rl} \text{Sum of the squares — product} & = a^2+b^2-ab \\ \text{Multiply by sum} & = a+b \\ \hline & a^3+ab^2-a^2b \\ & \quad -ab^2+a^2b+b^3 \\ \hline \text{Product,} & = a^3+b^3 \end{array}$$

So that we perceive that the sum of the squares wanting the product, when multiplied by the sum, produces the sum of the cubes. Let us now perform the division, first taking the one factor for a divisor, and then the other; and let us see whether in each case the result of the division will give us the other factor. First, dividing by the multiplier, we have this operation:—

$$\begin{array}{l} a+b)a^3+b^3\,(a^2-ab+b^2 \\ \quad -a^3+a^2b \\ \hline \qquad -a^2b+b^3 \\ \qquad +a^2b+ab^2 \\ \hline \qquad\qquad +ab^2+b^3 \\ \qquad\qquad -ab^2-b^3 \\ \hline \qquad\qquad\qquad 0. \end{array}$$

The first line of the above operation consists of the divisor, $a+b$, the dividend, a^3+b^3, and a place for the quotient as it is found. The divisor and dividend are placed with the letters in the same order, which is a matter of convenience though not one of necessity. Comparing a, the first term of the divisor, with a^3, the first of the dividend, we find that as $a^2 \times a$ is $= a^3$, the first term of the quotient must be a^2; and because the term of the divisor and that of the dividend have both the sign $+$, or, which is the same thing, are without any sign, a^2 in the quotient must be $+$.

We next multiply both terms of the divisor by a^2, and change the signs, which is the same as subtracting the products, and thus we get the second line $-a^3-a^2b$. Comparing the terms of this with those of the dividend we find a^3 with the sign $+$, and also with the sign $-$, which destroy each other, and we have remaining $-a^2b+b^3$, which is the third line of the operation; and we again compare its first term with the first term of the divisor, that is, we compare a with $-a^2b$. It is easy to see that a will be got $a\,b$ times in this term, and that the sign must be $-$, or that in order to convert the products into remainders we must make their signs the same as that of the divisor. Performing this multiplication we have $+a^2b+ab^2$, in which the first terms destroy each other, and there remains ab^2+b^3 for the fifth line of our operation. Comparing the first term of this with the first of the divisor we perceive that $+b^2$ in the quotient will, if multiplied by a in the divisor, and the sign changed, produce $-ab^2$, which extinguishes the first term. Multiplying both terms of the divisor by b^2, and changing the signs, we obtain $-ab^2-b^3$, which exterminates the whole dividend. Therefore our whole quotient is a^2-ab+b^2, which is exactly the same as our multiplicand, though the terms are not arranged in exactly the same order.

It will be seen from this operation that the process of dividing algebraically is so simple as to be merely mechanical; for at each step we have only to select such a term for the quotient as shall with the first term of the divisor produce the same combination of letters as that of the dividend, and shall have such a sign as when changed shall be opposite to that of the dividend. It is of no consequence whether any of the other terms are the same or not, because the changing of the signs of those parts obtained by multiplication converts them into remainders; and if the multiplication and change of the signs be rightly performed every step of the operation will lead to a true result, whatever may be the difference of its appearance.

Neither need we conclude that we have committed errors, though the product of a compound divisor and quotient do not amount to the same identical expression as the dividend; because we have already shown that equi-multiples and like parts of any divisor and dividend will all lead to exactly the same quotient.

Before we can thoroughly understand division, and those general relations which are founded upon its principle, or rather in which its principle consists, it is necessary to have recourse to some farther explanations, which can be more conveniently made in a new section; we shall therefore close this one by subjoining the operation for the above example, as divided by the other factor.

$$
\begin{array}{l}
a^2-ab+b^2)\,a^3+b^3\,(a+b \\
\qquad\qquad\quad \underline{-a^3+a^2b-ab^2} \\
\qquad\qquad\qquad\quad +a^2b-ab^2+b^3 \\
\qquad\qquad\qquad\quad \underline{-a^2b+ab^2-b^3} \\
\qquad\qquad\qquad\qquad\quad 0.
\end{array}
$$

SECTION VII.

NATURE AND MANAGEMENT OF FRACTIONS.

A FRACTION is a quantity viewed in its relation to some other quantity of the same kind which is considered as a whole; and as every case of division may be considered as reducible either exactly or to any degree of nicety that may be required to an expression in which the divisor is one, or, which is exactly the same in effect, every possible case of division may be conceived as consisting of equi-multiples of 1 and the quotient by the divisor, the simple expression of division by writing the dividend over the divisor, and separating them by a line, is also the general expression of a fraction. Thus $\frac{a}{b}$ is an expression for any fraction, in which the quantity b is understood to mean a whole, or the number 1, and a any quantity whatever, only it must be one of the same kind with b. If a and b were expressed arithmetically it would be necessary to express them both in the same unit, in order that the numbers might express the same relation as the values; and when general expressions are used it is necessary to understand them in this manner.

Perhaps the simplest notion we can have of the nature of a fraction is the arithmetical one, which supposes that the whole is divided into as many equal parts as the under term of the fraction expresses, while the value of the fraction consists in the number of those parts which the upper term expresses. Thus, in the expression $\frac{19}{20}$ the under number 20 shows that something considered as a whole is understood to be divided into 20 equal parts; and the upper number 19 shows that the value of this particular fraction is 19 of those parts. From this it follows that the value of the fraction does not depend upon the absolute numbers in which it is expressed, but upon the relation of those numbers to each other; and that each of the two numbers has a distinct operation to perform.

One whole, by whatever number it may be expressed, may be considered as always meaning the very same quantity, unless the contrary is expressly stated; and thus, the larger number which the under term of a fraction expresses, the smaller must be the value of every individual 1 of that number; but the larger the upper term, the value must always be the greater. A fraction may thus be considered as having a sort of double value, or a value which may at any rate be considered as the result of two operations, a division by the under term to find the value of 1 in the upper number, and a multiplication of the value so found by the upper number.

In an arithmetical point of view, the number of the under term fixes the denomination of the fraction, in the very same way as the denominations of real quantities are fixed by the standards in which they are counted; and for this reason the under number is called the *denominator* of the fraction. The denominator is thus, as it were, "small change" for the integer number 1, just as shillings are small change for a pound, or yards are small change for a mile. The upper number shows how much of this small change the fraction consists of, and for this reason it is called the *numerator*, or "the teller of the number" of the fraction. It may be any number, equal to the denominator, or greater, or less; and it may be a number which cannot be exactly expressed in terms of the denominator, at the same time that there is between the two a relation which we can perfectly understand.

Hence the doctrine of fractions is a very general one in mathematical science, as it involves all comparisons in which the whole value of one quantity is compared with the whole value of another. There is something neat in the signs which are used to express the comparisons. $a : b$ is relation generally, and says little more than that a and b are quantities of the same kind; $\frac{a}{b}$ is a more definite statement of the relation, for it points out that a is the standard with which b is compared. $a \div b$ with the compound sign is more definite still, for it points out the difference of the related quantities; but the line in it is not the sign — pointing out the difference by which the one quantity exceeds or falls short of the other, it relates to *the whole* of both, which is not necessary in the case of the mere difference as obtained by subtraction.

A fraction is still a quantity though the value of that quantity is expressed by a relation; and thus we must have some means of knowing when two fractions are equal and when not. We cannot tell this generally, or even in some common cases, by comparing the numerators; for these have equal values with equal expressions only when the denominators are also equal. If both consist of the same expressions their equality is of no use, as we can draw no conclusion from it.

Neither can we make the comparison generally if the terms are sums or differences indicated by the signs + or —. Thus we cannot tell whether $\frac{a+c}{b+d}$, or $\frac{a+b}{c-d}$, is the greater.

But if we have any means of showing that the two products arising from the multiplication of the numerator of each by the denominator of the other are equal, then we are in a condition for proving the equality of the two fractions; and this is important, as being the foundation of the rule of proportion, or "rule of three," which is so valuable in reasoning and calculation, both in arithmetic and in mathematics generally.

Now, from the connection that there is between a fraction and a case of division, it is evident that all equal fractions must have their terms equi-multiples of that form of the fraction which has 1 for its denominator. Thus, if $\frac{a}{b}=\frac{c}{d}$, then the product ad is equal to the product bc.

For, let the form of the fraction of which these terms are equi-multiples be $\frac{q}{1}$, and no matter whether q is less than 1, equal to 1, greater than 1, or whether it can or cannot be expressed in terms of 1, $\frac{a}{b}$ must be some multiple of $\frac{q}{1}$, and so must $\frac{c}{d}$. But m and n stand for any multipliers whatsoever. Therefore $\frac{a}{b}$ may be called $= \frac{q \times m}{1 \times m}$, that is, $\frac{qm}{m}$; and $\frac{c}{d} = \frac{q \times n}{1 \times n}$, that is, $\frac{qn}{n}$. Multiply qm by n, and qn by m, and we have $qmn = qnm$, which is an identical proportion, the product of the same three factors; and these factors are perfectly general, and may be anything, provided that those which are expressed by the same letters are equal to each other.

Take an example: a man is entitled to $\frac{13}{16}$ of a pound, would 16 shillings pay him? 16 shillings is $\frac{16}{20}$; and the question is, is it equal to $\frac{13}{16}$? Multiply the numerator of each by the denominator of the other, and we obtain for the first $16 \times 16 = 256$, and for the second $13 \times 20 = 260$, which is more than the other, so that 16 shillings is not quite enough.

Let us see how much it wants. The numerator of each fraction has been multiplied by the denominator of the other; and if we multiply the denominator by the same we shall have equimultiples, or fractions of the same value as the original ones, and they will at the same time have equal denominations; that is,

$$\frac{13 \times 20}{16 \times 20} = \frac{260}{320}, \text{ and } \frac{16 \times 16}{20 \times 16} = \frac{256}{320}.$$

The difference of these, or $\frac{260}{320} - \frac{256}{320}$, is $= \frac{4}{320}$, which, it is easy to see, is $= \frac{1}{80}$ of a pound, or $\frac{1}{4}$ of a shilling, or 3 pence more than 16 shillings the man would require.

Here we have discovered not only how fractions may be compared so as to ascertain which is the greater, but also how they may be prepared for addition and subtraction. We have only to multiply the terms of each by the denominators of all the others; and they will be all equi-multiples, with equal denominators.

Thus, in the fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, and $\frac{g}{h}$, we have $\frac{a}{b} = \frac{adfh}{bdfh}$, $\frac{c}{d} = \frac{abfh}{bdfh}$, $\frac{e}{f} = \frac{ebdh}{bfdh}$, and $\frac{g}{h} = \frac{gbdf}{bdfh}$; which are all equi-multiples, having a common denominator, and their sum is $= \frac{adfh + cbfh + ebdh + gbdf}{bdfh}$.

But, in order that we may be able to manage fractions with ease and certainty, we must consider how a fraction which presents itself in a complicated form may be made as simple as possible without altering its value.

Now, we cannot have any general means of simplifying the terms of fractions by addition or subtraction; because both the sum and the difference of the terms of two equal fractions are still the same fraction. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+c}{b+d}$ and also $\frac{a-c}{b-d}$ are equal to each other, and also to $\frac{a}{b}$ or $\frac{c}{d}$; for, multiplying the numerator of each by the denominator of the other, we have $(a+c) \times (b-d) = ab + bc - ad - cd$, and $(a-c) \times b+d = ab - bc + ad - cd$, which are evidently equal to each other, for all the four simple products in both are equal, and two have the sign + and the other two the sign − in each; and thus, though they express accurately the real ratio of $a : b$ or $c : d$ multiplied by the terms of the other equal ratio reversed, they are each equal to nothing.

As multiplying the one term and dividing the other by the same quantity have exactly the same effect on the value of a fraction, we can shift a multiplier or divisor from the one to the other at pleasure, provided that it affects the whole of the term in which it at first appears. Thus $\frac{a}{b}$, which is $\frac{a}{1 \times b}$, may, by changing the multiplier b of the denominator to a divisor of the numerator, be changed to $\frac{\frac{a}{b}}{1}$, or $\frac{7}{8}$ may be made $\frac{\frac{7}{8}}{1}$ without altering the value. Generally speaking, this renders the fraction more complicated, but there are cases in which it is of use.

The converse is much more useful, for by means of it all division of fractions may be changed into multiplication. Thus $\frac{\frac{a}{b}}{\frac{c}{d}}$ is evidently an expression for the division of $\frac{a}{b}$ by $\frac{c}{d}$; and making the divisor of each term a multiplier of the other, we have $\frac{ad}{bc}$, which is the terms of the dividend or numerator multiplied by those of the divisor or denominator, inverted; from which we may derive this general rule for dividing one fraction by another: turn the divisor upside down and multiply.

As the numerators of fractions are multipliers, and the denominators divisors, it follows that fractions are multiplied by multiplying their numerators for numerator, and their denominators for denominator. Thus $\frac{a}{b} \times \frac{c}{d}$, $\frac{a \times c}{b \times d}$, and $\frac{ac}{bd}$ all have the same meaning.

Any quantity may be changed into the form of a fraction by writing 1 for its denominator. Thus 3 may be $\frac{3}{1}$, and generally a may be $\frac{a}{1}$; and, from what has been already said, the quantity, whether expressed by a number, or a letter, or combination of letters, is a multiplier, but it may be changed into a divisor by inverting its terms. Thus $\frac{3}{1}$ is 3 as a multiplier, $\frac{1}{3}$ is 3 as a divisor, and generally $\frac{a}{1}$ is a multiplier, and $\frac{1}{a}$ a divisor.

It is evident that $\frac{a}{1}$ is any quantity divided by the number 1, which is just that quantity itself; and that $\frac{1}{a}$ is the number 1 divided by any quantity a, and as the first of these is the expression for the quantity as a multiplier, and the second the expression for the same quantity as a divisor, they are the opposites of each other; and for this reason 1 divided by any quantity is called the *reciprocal* of that quantity, and that division by any quantity is the same as multiplication by its reciprocal. We need hardly repeat that the reciprocal of a fraction is that fraction with its terms inverted. Thus, if $\frac{a}{b}$ is any fraction, $\frac{b}{a}$ is the reciprocal of that fraction.

Here it may be proper to inquire into what is the difference between any quantity and its reciprocal, as, for instance, what is the difference between $\frac{a}{b}$ and $\frac{b}{a}$? or, which is the same thing, what is the value of $\frac{a}{b} - \frac{b}{a}$? According to the principle formerly explained, we must multiply the numerator of each by the denominator of the other for the respective numerators, and the two denominators together for the common denominator, which gives us $\frac{a}{b} = \frac{a^2}{a\,b}$, and $\frac{b}{a} = \frac{b^2}{a\,b}$; and consequently the difference $= \frac{a^2 - b^2}{a\,b}$, that is, the difference of the squares divided by the product. Thus, the difference between $\frac{4}{3}$ and $\frac{3}{4}$ is $= \frac{4^2 - 3^2}{4 \times 3} = \frac{16 - 9}{12} = \frac{7}{12}$; the difference between $\frac{4}{1}$ and $\frac{1}{4}$, that is, 4, and the reciprocal of 4, is $\frac{16-1}{4} = \frac{15}{4} = 3\frac{3}{4}$.

The product of any quantity by its reciprocal is always $= 1$, for it is always a fraction of which the numerator and denominator are equal.

The quotient by the reciprocal is the square of the quantity $\frac{a}{1} \div \frac{1}{a}$ is $\frac{a \times a}{1 \times 1} = a^2$; $\frac{a}{b} \div \frac{b}{a}$ is $\frac{a \times a}{b \times b} = \frac{a^2}{b^2}$.

When quantities are stated as the terms of ratios, and not as fractions, the reciprocal is the terms transposed; thus, $b : a$ is the reciprocal of $a : b$.

If two quantities are mutually the reciprocals of each other, any equi-multiple of either of them must also be the reciprocal of any equi-multiple of the other. Thus, $\frac{m\,a}{m\,b}$ and $\frac{n\,b}{n\,a}$ are reciprocals, and a and b stand for any quantities whatever, and m and n for any multipliers. Hence,

"If two quantities have the same proportion as other two, the product of the first and fourth, when they are placed in the order of proportion, must be equal to that of the second and third;" that is,

$$\text{If } a : b = c : d, \text{ or } a : b :: c : d, \text{ then, } a\,d = b\,c;$$

for since $a : b = c : d$, that is, $\frac{a}{b} = \frac{c}{d}$, $\frac{d}{c}$ is the reciprocal of $\frac{a}{b}$, and $\frac{a \times d}{b \times c} = 1$; therefore $a\,d = b\,c$.

The principles which have been stated contain the foundation of the management of common fractional quantities, whether expressed by numbers or by letters, and we can better explain the method of treating exponential fractions in another section. It may not be amiss, however, to recapitulate the leading points.

1. Fractions are not, arithmetically, quantities of the same kind, unless they have the same denominators; but they can always be reduced to a common denominator by multiplying both terms of each by all the denominators, except its own, and then the sums or differences of the numerators may be found in the same way as in quantities not fractional.—The denominators are not subjects of addition or of subtraction.

As the numerator is always a multiplier, and the denominator a divisor, it follows that, to multiply any number of fractions together, we have only to multiply all the numerators for numerator, and all the denominators for denominator; and if there be any quantities which are not fractions among the factors, they may be put in a fractional form by writing 1 for the denominator of each. To divide fractions, we have only to turn the divisors into their reciprocals, which is done by inverting the terms, and then treat them as in multiplication.

When there is a number of multiplications and divisions to be performed, it is often very convenient to throw them into a general fraction, by placing all the multipliers above a line, and all the divisors below the same, and connecting them by the sign × repeated between every two. Thus, for instance, if it were required to divide the product of 12, 16, 9, 28, and 7, by the product of 49, 27, 6, and 4, we might arrange them thus,—

$$\frac{12 \times 16 \times 9 \times 28 \times 7}{49 \times 27 \times 6 \times 4} = \frac{2 \times 4 \times 4}{3} = \frac{32}{3} = 10\frac{2}{3}$$

Here we leave out all the factors which are common to the two terms; and divide the product of the remaining ones above the line, by the product of those below. This is one of the most useful operations in arithmetic.

It will be seen at once that this result is obtained by throwing out those factors which are common to both terms of the original fraction, and by this means the greater part of the labour of multiplying and dividing is saved. This is more a matter of convenience in practice, than of investigation of principle; but still it is so useful, that it is very desirable that every one who wishes to be an expert calculator, even in common matters, should be so well acquainted with what numbers consist of equal factors, and what do not, as to see at once how the expression may be shortened; we shall, therefore, make this the subject of the next section.

SECTION VII.

THE FACTORS, THE DIVISORS OR MEASURES, AND THE MULTIPLES OF NUMBERS.

From what has been shown in former sections, it will easily be understood that, regarding them merely as numbers, and without reference to their standing either for one kind of quantities or for another, there is a very great difference between numbers considered simply as numbers or answers to the question "How many?" and numbers considered as multipliers or divisors.

In numbers simply considered, 1 is the standard of value; it always counts; and the symbol of no value is 0; but in a multiplier or a divisor, 1 is the standard of no value, and 0 has a very different signification. As a multiplier, 0 points out, not that there shall be no multiplication, but that there shall be no multiplicand; and 1 is really the sign of no multiplication. As a divisor, 1 is the sign that there shall be no division, and 0 is a sign that, whether the dividend be small or great, the quotient shall be infinite—shall be all possible numbers; and if we are to write it down, we may write any number whatever with equal propriety. $n \times o$ (meaning by n the greatest possible number that any one can think of) is $= 0$, but the product, divided by the multiplier, gives the multiplicand; therefore $\frac{a}{0}$ is infinitely great, and when it occurs, it is usually expressed by a double 0 laid horizontally, ∞.

A number which cannot be divided without remainder by any number except 1 and itself (which is no division), is called a *prime number;* it is an original number, or one which is not the result of any operation.

A number which can be divided without remainder is called a *composite* number, because it may be said to be composed of either the quotient or the divisor, repeated as many times as the other expresses. A composite number has, then, always two *divisors;* and as it is composed of the product of these divisions, they are called the *factors* of it. Thus the factors and divisors of a number always mean the very same numbers. Still it is necessary to distinguish between them, because, when we have the factors given, whatever may be their number or value, we can in all cases find their product; but when we have a product given, we have no general means of finding what its factors may be, or whether it is a product at all.

In Algebra, where the operations are expressed as well as the quantities, this difficulty is not felt; there are particular cases in which we can get the better of it in arithmetic, and no one can be expert, even as a common accountant, without being able to perceive those cases where they occur.

The natural numbers taken in their order, 1, 2, 3, 4, &c., form a *series* or succession, beginning at 1, and increasing by the addition of 1, as a common difference; and the problem is to determine what terms of this series are prime, and what are composite.

1 is evidently a prime number, and so is 2; but we can see that 2 must be a factor of every *second* number after this—of 4, 6, 8, 10, 12, &c., but not of any other number. 2 is thus the smallest factor which any number can have, and the other factor corresponding to it must be half the number. Hence we are sure that half the series of the natural numbers are composite, and that no factor of a number can be greater than one-half of it.

This is not much, but it is a beginning, and we may see what more we can make of it before we proceed farther. The numbers of which 2 is not a factor, always have an odd 1 when we attempt to divide them by 2; hence we call them *odd numbers*, and those of which 2 is a factor are *even numbers*.

If we divide ever so many even numbers by 2, there is not an odd 1; therefore the sum of any number of even numbers is an even number; so also is the sum of an even number of odd numbers, for the odd 1s make an even number, and when they are taken away the other numbers are all even; but if the number of odd numbers is odd, there is an odd 1, which makes the sum odd.

Here we find a principle of some importance. We have seen that, if 2 divide the sum of the remainders, it will divide the sum of the numbers; and this is general, applying to any numbers and any divisor, for the single numbers are all divisible, except the remainders; and if the sum of the remainders is divisible, so must the sum of the numbers. Hence,

If a number is a factor of the remainders, it is a factor of the sum; and a factor of two numbers must be a factor of their sum, their difference, their product, or any multiple of each or all of them.

To return to the natural numbers;—3 is a prime number, but 4 is not; it is the second after 2, and an even number. 5 is prime, because 4 is the only composite number below it, and $5=4+1$; and no factor of 4 can be a factor of 1. Hence no number and the one immediately following it in the scale can have a common factor.

It is evident that every second number is divisible by 2, every third one by 3, every fourth one by 4, every fifth by 5, and so on; also that the factors of every composite number will both fall in that place of the series which answers to their numbers multiplied, and that each will then have occurred in the series as often as the other expresses. Thus 4 and 5 occur together for the first time at 20, and it is the fifth occurrence of 4, and the fourth of 5.

As 1 is the measure of all numbers, whether prime or composite, it follows that all numbers whatever must fall together at some place of the series, and that there can be prime numbers only at those places where no number falls. Upon this principle we could, by mere mechanical labour, construct a table of all prime numbers, and of all divisors of composite ones, as far as we chose; and there are few better exercises for a beginner in the study of numbers, than the construction of such a table. Of course it is not necessary to go beyond half the series, as no divisor of a number can be greater than the half. The following is the arrangement as far as 20:—

PRIME NUMBERS AND FACTORS.

1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16,	17,	18,	19,	20
P.	P.	P.	2	P.	2	P.	2	3	2	P.	2	P.	2	3	2	P.	2	P.	2
					3		4		5		3		7	5	4		3		4
											4				8		6		5
											6						9		10

From this example it will be seen that the factors always occur in pairs, and that if there are more than one pair, two of the divisors are composite numbers; and that when there is a 2 among the factors, there is always another factor equal to half the number.

If the places of the prime numbers, which are marked by the letter P, are examined, it will be found that they are either immediately before or immediately after 2 or 3, or some number divisible both by 2 and by 3, that is, by 6. Therefore we have this general principle:—every prime number, either with 1 added to it, or 1 subtracted from it, must be divisible by 6; but we are unable to state the converse of this as true; for every number which +1 or —1 is divisible by 6, is not prime: and, farther than this, numbers which have not this property, and are not divisible by 6, without addition or subtraction, are always even numbers. Hence we are reduced to merely making trial, when we get an odd number, unless in a few particular cases, and in these our assistance is derived from the scale of numbers, and, except in one or two of them, the labour of investigating a rule is greater than that of ascertaining the fact by trial, and just as much confined to the particular case.

1. If we can show that any factor will divide a number, except a certain number of figures on the right, whatever the figures to the left of these may be; and if we perceive, from the particular number before us, that this factor will divide those right-hand figures, then it will follow, on the principle that a factor of all the particular numbers is a factor of the sum, that the factor in question is a factor of the number. Now it was shown when treating of the scale of numbers, that any integer number may be considered as consisting of as many numbers as there are figures in it; that it is all 10s except the right-hand figure, all 100s except two figures, all 1000s except three, and so on; therefore, if any factor of 10 divide the unit's figure, it must divide the number; if any factor of 100 divide the units and tens, it must divide the number; if any factor of 1000 divide the units, tens, and hundreds, it must divide the number, and so on.

The factors of 10 are 5 and 2, and each of them occurs twice as a factor in 100, three times in 1000; so that we may express the places in a number by repeating 5×2 for every place; as, for instance, 1000 might be written $(5 \times 2) \times (5 \times 2) \times (5 \times 2)$. Now these may be compounded into any two factors which make 1000 when multiplied; as $(5\times5\times5)\times(2\times2\times2)$, that is, 125×8; and if the units, tens, and hundreds be divisible either by 125 or by 8, they must also be divisible by all products of the factors of these numbers. But if 2 is a factor along with any number of 5s as factors, it will produce 0 in the units; if 4, it will produce 0 0 in the units and tens; therefore we may state that, if the units be divisible by 5, the units and tens by 25, or the units, tens, and hundreds by 125, each of these will in that case be a factor, or divisor of the whole number, whatever may be the figures to the left; and generally that, if any number of figures on the right hand be divisible by the continued product of as many 5s as there are figures, the whole number will be divisible by the same.

There is a very convenient method of shortening numbers by this means: if the unit figure is 5, multiplying by 2 will clear it away; if two figures are divisible by 25, multiplying by 4 will clear them away; if three figures are divisible by 125, multiplying by 8 will clear them away, and always, as another figure is divisible by another 5 as a factor, another 2 as a factor of the multiplier will clear it away. This method is often useful in the management of decimal numbers.

2. If the sum of the figures in any number is divisible by 9, then 9 is a factor of the number; and if 9 is a factor, 3 is a factor twice over; also 3 is a factor if the sum of the figures is divisible by 3. The principle upon which this depends is a very obvious one:—if from any number of 10s there is taken an equal number of 9s, the same number of 1s must remain. Now every figure in a number, except the units, is a number of 10s, and therefore, if every figure is divided by 9 down to the ten's place, there will remain the same number of units as the figure expresses; thus we may consider all the figures as remainders, after dividing by 9; and it has been shown that, if the divisor

divides the sum of the remainders, it will divide the sum of the numbers.

If the sum of the figures, counting from the unit's place inclusive, in the odd places, be equal to that of all those in the even places, 11 will divide the number. This also is easily shown, for any number of 10s wants an equal number of 1s of being an equal number of 11s; so that, whatever the figure is which has 0 annexed to it, in order to be divided by 11, the remainder added to the figure will always make 11, only where two 0s have to be added, these will count only as one figure, and the quotient will be 09. Thus, 1243 is divisible by 11. The first figure is 1000, which gives 90 for the quotient, and 10 over; the second is 200, which gives 18, and 2 over; the third is 40, and gives 3, and 7 over; and the fourth is 3 over. Thus all the number is divisible by 11, except 10 + 2 + 7 + 3 = 22, which also is divisible by 11, and consequently the whole of the number.

From this peculiarity of numbers divisible by 11, we can obtain the quotient in rather a curious way, by beginning at the right, and writing twice over any figure that will make the last of two = 0, only if we add 10 to the upper figure, we must add 1 to the next under one. Thus, 1825406 is divisible by 11, and the quotient may be found thus:—

```
1825406
     66
    44
   99
  55
 77
11        = 175946, quotient.
```

Or we might do it by the first figure of each pair, thus:—

```
1825406
      6
     4
    9
   5
  7
 1        =175946.
```

These are the only cases in which finding the factors or divisors of one number will repay the trouble of the investigation, unless we continue 3 or 9 and 11. If the sum of the figures divides by 3, and the even places equal the odd, the number divides by 33; and if the sum of the figures divides by 9, and the even and odd are equal, the number divides by 99. Thus, the number 126522 divides by 99.

The factors of a single number are of comparatively small importance; but the common factors or common divisors of two or more numbers are very useful, and, fortunately, there is a very simple general method of finding the greatest common factor, or, as it is called, the greatest *common measure* of two numbers.

It has already been shown that a common factor divides the sum, the difference, and any multiple; and if that factor and a third number again have a common factor, that factor will divide all the three numbers. Thus, using the last-found factor and another number, we may proceed to as many as we please.

The clearest way of showing this will be by an example in numbers. Let it, then, be required to find the greatest common divisor, or measure of 8172 and 6354.

1. Divide the greater by the less.

```
6354 ) 8172 ( 1, quotient.
       6354
       ----
       1818 remains ;
```

therefore 6354 is not a divisor; but 1818 is the difference of the proposed numbers, and must be divisible by all their divisors, and so must every multiple of it, and also the difference between it, or any of its multiples, and 6354; therefore,

2. Divide the divisor by the remainder,

```
1818 ) 6354 ( 3, quotient.
       5454
       ----
        900 remains;
```

therefore 1818 is not a divisor.

3. Divide again—

```
900 ) 1818 ( 2, quotient.
      1800
      ----
        18 remains;
```

therefore 900 is not a divisor.

4. Divide yet again—

```
18 ) 900 ( 50, quotient.
     900
     ---
       0 remains;
```

therefore 18 is a divisor, for it divides 900, which is 50×18, and 1800, which is 900×2, and 1818, which is $1800 + 18$, and 5454, which is 1818×3, and 6354, which is $5454 + 900$, and 8172, which is $6354 + 1818$; consequently 18 is a common factor, or measure, or divisor of 8172 and 6354; and it is the greatest number that can be a measure of them both; for any number which is so must be a measure of 18, and it is evident that 18 is the greatest number that can be so. Generalising this operation, we have the common rule for finding the greatest common measure of any two numbers :—divide the greater by the less, and the divisor by the remainder continually till nothing remains, and the last divisor is the greatest common measure.

To reduce any fraction, or any ratio to its lowest terms, we have only to find the greatest common measure and divide both terms by it: thus, if our example were the ratio 6354 : 8172, or the fraction $\frac{6354}{8172}$, the lowest term would be 353 : 454, or the fraction $\frac{353}{454}$; and if they are tried by the same operation it will be found that these numbers have no common divisor.

Numbers which have no common divisor are said to be *prime to each other;* and such numbers taken singly may be either prime or composite.

Any number of which several other numbers are factors or divisors is called a *common multiple* of them, and the least number of which they all are divisors is their *least common multiple.* But, before we examine the multiples of numbers, it will be of use to revert to the process by which we find the greatest common divisor, because that process is useful in practice, even though the result of it should be that the numbers have no common divisor.

If we express the first step of the division in the former example we have $8127 \times 6354 = \frac{1}{1\frac{1818}{6354}}$; if we substitute the second division for the fraction in this denominator, we have

$\frac{1}{3\frac{900}{1818}}$; if we substitute the fraction for the division, in this we have $\frac{1}{2\frac{18}{900}}$; and from this, by the last division, we have $\frac{1}{50}$. But the simple fraction in each of these belongs to the denominator of the fraction before it; therefore, the whole assumes this form :—

$$\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{50.}}}}$$

This is called a *continued fraction*, because every following fraction is part of the denominator of the one before it.

We may take the whole of this fraction and reduce it, which will give us the lowest terms of the fraction $\frac{6354}{8172}$, that is, $\frac{353}{454}$. The last two parts are $\frac{1}{2\frac{1}{50}}$; and multiplying both terms of this by 50, to clear it of the fraction in the denominator, we have $\frac{1\times 50}{2\times 50+1}$, which, performing the multiplications and the addition, gives $\frac{50}{101}$. Substitute this in the term above, and it becomes $\frac{1}{3\frac{50}{101}}$; reduce this and there is $\frac{1\times 101}{3\times 101+50}$, and performing the operations, we have $\frac{101}{353}$. Substituting this for its value, we have $\frac{1}{1\frac{101}{353}}$; and reducing this, $\frac{1\times 353}{1\times 353+101}=\frac{353}{454}$, the same as we obtained by the direct division of the terms by their greatest common measure.

But we might not have occasion to make use of all the nicety of the lowest terms of the fraction, or the ratio (for it is the same in either case), and then the continued fraction furnishes us with a series of approximations by taking one, two, three, or more terms, at pleasure, from the beginning.

In the above fraction, the first term gives us $\frac{1}{1}$, or that 353 = 454, which is by much too high; the first and second terms, $\frac{1}{1\frac{1}{3}}$ give us $\frac{1 \times 3}{1 \times 3 + 1} = \frac{3}{4}$. The first, second, and third terms, $\cfrac{1}{1\cfrac{1}{3\frac{1}{2}}}$ give us $\cfrac{1}{1\cfrac{1 \times 2}{3 \times 2 + 1}} = \cfrac{1}{1\frac{2}{7}} = \frac{7}{9}$; and the whole terms give us $\frac{353}{454}$, as before. So that we have the series $\frac{1}{1}$, $\frac{3}{4}$, $\frac{7}{9}$, $\frac{353}{454}$, each nearer the truth than the one before it, till we come to the last, which takes in all the terms, and is, in consequence, exactly true.

But if we examine the way in which these terms are obtained, we find that the second is the first multiplied by the second quotient, with 1 added to the product of the denominator; and that each succeeding one is the one before it multiplied by the next quotient, and the one before that added to the product, as in this operation:—

Quotients .	1,	3,	2,	50.
Statements .	$\frac{1}{1}$,	$\frac{1 \times 3}{1 \times 3 + 1}$,	$\frac{3 \times 2 + 1}{4 \times 2 + 1}$,	$\frac{7 \times 50 + 3}{9 \times 50 + 4}$.
Fractions .	1,	$\frac{3}{4}$,	$\frac{7}{9}$,	$\frac{353}{454}$.

If we compare the statements we can see that the difference of the values of every two adjoining ones is 1 divided by the product of the denominators, thus $\frac{1}{1}$ is $\frac{1}{4}$ greater than $\frac{3}{4}$, $\frac{3}{4}$ is $\frac{1}{36}$ less than $\frac{7}{9}$, and $\frac{7}{9}$ is $\frac{1}{4086}$ greater than $\frac{353}{454}$. Thus, if we take the smaller of the given numbers as the numerator of the fraction, or as the term compared with the standard in the ratio, the first fraction gives the value too high, the second too low, the third too high, and so on alternately, till we come to the truth in those cases which terminate, or without limit in those which do not. Each is thus nearer the truth than the difference between it and the next.

Let us compare these by reducing them to a common denominator, that is, by multiplying the terms of each by all the denominators except its own, thus:—

$$\frac{1}{1} \times 4 \times 9 \times 454 = \frac{16344}{16344}, \text{ is } \frac{3536}{16344} \text{ too much.}$$

$$\frac{3}{4} \times 9 \times 454 = \frac{12258}{16344}, \text{ is } \frac{450}{16344} \text{ too little.}$$

$$\frac{7}{9} \times 4 \times 464 = \frac{12712}{16344}, \text{ is } \frac{4}{16344} \text{ too much.}$$

$$\frac{353}{454} \times 9 \times 4 \frac{12708}{16344} \text{ is the truth.}$$

$\frac{7}{9}$ is very near the truth, being only $\frac{1}{4084}$ of 1 from it, which is so little, that for any common purpose 7 and 9 would do just as well as 353 and 454. We shall have occasion to take some further notice of continued fractions, on account of the assistance they give us in matters much more difficult than the present.

If it is borne in mind that the least common multiple of any number of numbers is the smallest number that can be divided by each of them without remainder, and that when they are all prime to each other this number is the product of them all, the following practical directions will be understood without any explanation:—write the numbers in a line after each other, divide them by their common factors or measures till no number can be found that will divide two of them; then multiply all the divisors and undivided numbers, and the product will be the least common multiple.

Let it be required to find the least common multiple of 18, 24, 16, 15, 14, and 9. Arrange them

$$18, 24, 16, 15, 14, 9;$$

2 divides them all but 15 and 9, and the results are,

$$9, 12, 8, 15, 7, 9 \times 2.$$

3 divides them all except 7 and 8, and the results are,

$$3, 4, 8, 5, 7, 3 \times 2 \times 3;$$

3 divides 3 and 3, and the results are,

$$1, 4, 8, 5, 7, 1 \times 2 \times 3 \times 3.$$

4 divides 4 and 8, and the results, leaving out the 1s, which make nothing as multipliers, are,

$$2, 5, 7 \times 2 \times 3 \times 3 \times 4,$$

which are not only prime to each other, but all prime numbers except the divisors; therefore,

$$2 \times 5 \times 7 \times 2 \times 3 \times 4 = \text{least common multiple.}$$

In such cases we can often get the product with very little trouble. In the above $2 \times 5 \times 2 \times 4 = 80$, $80 \times 7 = 560$, $560 \times 9(3 \times 3) = 5040$.

The continual product is a much larger number, being

$$18 \times 24 \times 16 \times 15 \times 14 \times 9 = 130603680,$$

which is 2592 times greater than the least common multiple, and would greatly increase the labour and chance of error in any calculation into which it enters.

One of the neatest applications of the least common multiple is the reducing of fractions to a common denominator, in order to add them; and the least common multiple is, of course, the least common denominator.

Calling the numerator n, the denominator d, and the least common multiple m, the formula is $\frac{n\,m}{d}$. But in all cases where we have a multiplication and a division to perform, and know that we can divide one of the factors without remainder, we abridge our labour very considerably by performing the division first, and thus converting the quotient into a multiplier. Now as the least common multiple is necessarily divisible by all the denominators, we can divide it and obtain a multiplier for each numerator. The formula will thus become $n \times \frac{m}{d}$ the division of m by d being performed, and the quotient used as a multiplier.

Let us illustrate this by an example:—

Uncle Nathan, who by great skill in calculating the arbitration of exchanges, and various little other arts which are well known on the Stock Exchange in the city of London, at which his mind had been so constantly, so silently, and so cautiously at work for half a century, that he got the name of "the calculating clock with the dead beat 'scapement," was in the fulness of time, and the abundance of his accumulations, gathered to his fathers. He left a goodly fortune; but as part of it was in the hands of half the kings of the world, he could not tell its amount, and therefore could not bequeath it to his six nephews in specific sums; therefore he devised it fractionally as follows:— $\frac{13}{50}$ to Gripe, $\frac{6}{25}$ to Grind'em, $\frac{11}{64}$ to Grub, $\frac{5}{32}$ to Grudge, $\frac{7}{64}$ to Grippy, and $\frac{3}{50}$ to Gad, who, being something of a wandering blade, stood lowest in the favour of Uncle Nathan. Furthermore, he willed that Goosewing, who had been his trusty and well-beloved scout and scribe for many years, should arrange the whole matter, transfer to each of the six nephews his legacy, and take 2500*l.* for his trouble. It is required to find the amount of Uncle Nathan's savings, and the portion which came to each of the nephews.

It is clear that the fortune, whatever it is, is 1, and may be represented by any fraction of which the numerator and denominator are exactly equal; and it is also clear that the fortune may likewise be represented by all the fractions which the nephews are to receive, together with Goosewing's 2500*l.*; therefore,

$$\frac{13}{50}+\frac{6}{25}+\frac{11}{64}+\frac{5}{32}+\frac{7}{64}+\frac{3}{50}+2500l. = 1.$$

We shall, in the mean time, call the 2500*l.* a; and our first business will be to collect all the fractions into one sum, which $+a$ will give us the fortune, or at least the proportion which a bears to the other shares, and then from that we can easily get the shares themselves. In order to do this let us first find the least common multiple of the denominators,

50, 25, 64, 32, 64, 50.

Here we see at once that 50×64 can be divided by all these numbers; therefore,

$$64 \times 50 = 3200 = \text{least common multiple.}$$

This least common multiple is the common denominator of the fractions; and to find the numerator of each in terms of this denominator we have merely to divide it by each denominator, and multiply the respective numerator by the quotient. For this we have

1. Gripe . . $\frac{3200}{50} = 64$, Gripe's multiplier

2. Grind'em . $\frac{3200}{25} = 128$, Grind'em's multiplier

3. Grub . . $\frac{3200}{64} = 50$, Grub's multiplier

4. Grudge . . $\frac{3200}{32} = 100$, Grudge's multiplier

5. Grippy . $\frac{3200}{64} = 50$, Grippy's multiplier

6. Gad . . $\frac{3200}{50} = 64$, Gad's multiplier.

We have next to multiply each one's share by his multiplier, and we get the proportional shares in terms of the denominator 3200,

1. Gripe . . $13 \times 64 = 832$, Gripe's number
2. Grind'em . $6 \times 128 = 768$, Grind'em's number
3. Grub . . $11 \times 50 = 550$, Grub's number
4. Grudge . $5 \times 100 = 500$, Grudge's number
5. Grippy . $7 \times 50 = 350$, Grippy's number
6. Gad . . $3 \times 64 = 192$, Gad's number

Sum of the numbers 3192 = numerator,

Thus the shares of the six nephews amount altogether to

$\frac{3192}{3200}$; hence we have,

$$\frac{3192}{3200} + a = 1.$$

The value of a is evidently the difference between the numerator and denominator of this fraction. If the numerator were the greater it would be —, and if the terms were equal it would be 0; but the numerator is less than the denominator, therefore Goosewing's fee is $\frac{8}{3200}$; or, disregarding the denominator, as it is the same in all, his share is 8, and the money value of it, according to the will, is 2500*l.*

If we divide 2500 by 8 we get 312*l.* 10*s.* as the value of every unit in the numbers; and this, multiplied by the proportional numbers, will give the shares, the sum of which, together with the scribe's fee, will be the whole fortune.

312*l.* 10*s.* is, changing the 10*s.* to a decimal, 312·5; wherefore,

1. Gripe	832 × 312·5 =	£260,000 to Gripe,
2. Grind'em . .	768 × 312·5 =	240,000 to Grind'em,
3. Grub	550 × 312·5 =	171,875 to Grub,
4. Grudge . . .	500 × 312·5 =	156,250 to Grudge,
5. Grippy . . .	350 × 312·5 =	109,375 to Grippy,
6. Gad	192 × 312·5 =	60,000 to Gad,
Goosewing . .	8 × 312·5 =	9,500 Goosewing's fee,
	Total fortune .	£1,000,000

We have given this example chiefly on account of its simplicity, and the consequent ease with which a reader not much conversant with figures can understand it; and having done so, we shall proceed, in the next section, very shortly to examine those principles in decimals which are most generally useful.

SECTION VIII.

SOME PROPERTIES OF DECIMALS.

In a former part of this work we pointed out the general nature of decimals: such as, that they are a continuation of the scale of numbers below the last place of integers, or the place of units, and that the several places in a decimal number bear exactly the same relation to each other as the places in an integer number—that is to say, that 1 in any place is equal to 10 in the place immediately to the right of it, to 100 two places to the right, and so on; that decimal numbers are added, subtracted, multiplied, and divided, exactly in the same manner as integer numbers; that the removing of the decimal point any number of places to the right is equivalent to the multiplying of the number as often successively by 10 as the number of places which the point is so removed; and that the removing of the decimal point in any number of places toward the left is equivalent to the dividing of the number as often by 10 as the number of places that the point is removed. It will also appear evident, from what has been said respecting fractions, that the denominator of a decimal number must consist of 1 with as many 0s annexed as there are figures in the decimal; and as the number of 0s is always equal to the number of times that 10 is a factor, if the decimal consists of n figures, the expression for the denominator will be 10^n.

The ready use of the decimals enables us, however, so much to simplify all the common applications of arithmetic to the business of life, and is so indispensable whenever the relations of magnitude enter into those calculations, that it is necessary for every one to understand their nature more thoroughly than it can be understood, without a knowledge not only of arithmetical fractions, but of the general relations of fractional quantities, as considered and expressed algebraically. We shall therefore resume the subject, in a very short section, as preparatory for understanding the nature and use of exponential numbers, and of logarithms.

If a decimal number is given, in which state it is presumed to express a complete value, its denominator is always known, being, as we have already said, 10^n, n being the number of figures, whether the significant ones begin at the decimal point or are preceded by any number of 0s. But decimal numbers do not present themselves to us naturally in the practice of calculation, unless in cases of division where the divisor is a power of 10, that is, 1 with 0s attached; therefore the cases where we are called upon to express numerical values in decimals are those in which we have to divide a less number by a greater, and this, of course, may occur either when the original dividend is less than the divisor, or when, after we have found the quotient in integers, as far as it can be obtained, there is still some remainder left.

As this remainder and the divisor are integer numbers, or, which amounts to the same thing, numbers expressed in the same place of the scale, it is evident that the largest possible remainder which can be left is 1 less than the divisor, and that the smallest remainder is 1. Thus, as the divisor is the denominator of the fraction, and the remainder the numerator, it is evident that the denominator —1 is the limit of the number of fractions having any denominator. Thus, if the denominator is 2, there can be but one fraction, $\frac{1}{2}$; if the denominator is 3, there can be but two fractions, $\frac{1}{3}$ and $\frac{2}{3}$; if it is 4, there can be but three fractions; and so on in all other cases.

The fraction which we meet with, or that which arises in any proposed case of division, may be any one of the numbers within the limits, that is, whatever may be the denominator the numerator may be 1, or denominator —1, or any number intermediate between them. Hence, if we take the general fraction $\frac{a}{b}$, in which a and b represent any numerator and denominator whatever, the decimal corresponding to that fraction, or expressing its value, will evidently be $\frac{n}{10^n}$; and the finding of the decimal will consist in determining the value of n, which represents not only as many places of decimals as there are units in the exponent ^{n}, but also the particular values of the figures in those places.

Now, from what was said of fractions in a former section, the general formula for changing a fraction from any denominator to an equal fraction having any other denominator, calling the proposed denominator d, is $\frac{ad}{b} = n$. But in the case of a decimal, $d = 10^n$; therefore the formula is $\frac{a10^n}{b}$. The multiplier 10^n may be used in successive factors, as $\frac{10a}{b}$ will give the first figure; and, using the remainder r in place of a, $\frac{10r}{b}$ will give another figure; and this operation may be continued as long as is necessary, the multiplying by 10 being nothing more than annexing 0 to the remainder; and for as many 0's as there are used there will be as many places of the decimal.

The original fraction, $\frac{a}{b}$, may be understood to be in its lowest terms, that is, that a and b have no common divisor; for if they have they may be divided by that divisor, and the lowest terms will thereby be obtained. Now when this is the case it is evident that, as a and b have no common divisor, b and $a10^n$ can have no common divisor, unless it is a divisor of 10^n. Nothing will divide a product without remainder except a factor of that product; and therefore no prime number will divide 10^n unless it is a factor of 10, because, how many multiplications soever may be expressed by the exponent ^{n}, none of them introduces any prime numbers, except the numbers 2 and 5, which are the factors of 10. Therefore a fraction which is in its lowest terms cannot be wholly reduced to a decimal if the denominator of it is any prime number, or contains as a factor any prime number, except 2 or 5. Decimals arising from prime denominators different from 2 and 5, or having factors different from those numbers, are called *interminate decimals;* and their denominators and 10^n are said to be prime to each other, that is, the one cannot be expressed in terms of the other by a single number, though the expression may be brought as near the truth as is necessary for any practical purpose; and we can, in all cases, after we have carried on the operation to a certain length, find the law of its continuance, and thus extend it to any length without more labour than that of writing down the figures.

The reason of this will readily appear from what has been stated as to the number of numerators which there can be to any one denominator. These, as we said, must always be 1 fewer than the number expressed by the denominator; and as, whatever the dividends may be, there cannot be more remainders in division than this number, it follows that every decimal obtained from a common fraction by adding 0s and dividing, must repeat some *series* or *circle* of figures, and that this circle of figures must always be 1 less than the number expressed by the product of all the prime factors of the denominator which are different from 2 and 5. Hence such decimals are, again, called *circulating* decimals.

There is some difference in the form of those decimals, according as the denominators from which they arise do or do not contain 2 or 5 as a factor, once or oftener, besides the prime factors different from 2 and 5; and as those which do not contain either 2 or 5 as a factor are the simplest in their form, we shall consider them first.

It will be easily perceived that, in order to ascertain how many figures any decimal will circulate, we have only to assume 1 as a numerator, and find how many 0s must be annexed to this 1, so that the last remainder may also be 1; and the number of 0s necessary for this purpose will be the number of circulating figures; also, if the quotient, or decimal with 1 for numerator, be ascertained for any denominator, we shall have only to multiply that decimal by any other numerator, in order to find the decimal answering to that numerator.

There is no general method hitherto discovered, or in all probability discoverable, by which this can be done. It might be extended to a certain length by a process similar to that by which we showed in a former section how the prime numbers may be discovered; but the method is more laborious, as the terms of comparison are the powers of 10, instead of the series of the natural numbers. We shall, however, mention one or two cases in which the number of figures is easily determined, not only because they are somewhat curious in themselves, but because they throw light upon some of the properties of numbers to which we may have occasion to advert afterwards.

In the first place, all fractions whose denominators are 1 less than any power of 10, circulate as many figures as the exponent of that power; and the figures which they do circulate are simply those of the numerator, without any further trouble than prefixing as many 0s as shall make them equal to the number of 9s in the denominator. Thus, $\frac{1}{9}$ is $=\cdot111$, &c.; $\frac{1}{99}$ is $=\cdot0101$, &c.; $\frac{1}{999}$ is $=\cdot001001$, &c.; and so in other cases. That this is true is apparent from mere inspection of the numbers, but it may be illustrated by actually performing the division. The value of any fraction is equal to the product of a fraction with the same denominator and 1 for numerator, multiplied by the numerator, as, for instance, $\frac{7}{9}$ is $=\frac{1}{9}\times 7$; and so on in all other cases. Hence we have this general principle:—if the denominator of a fraction consists of any number of 9s, the circulating decimal answering to that fraction will be the numerator, with 0s before it if necessary. Thus $\frac{13}{99}$ is = decimal $\cdot1313$, &c.; $\frac{6}{999}$ is = decimal $\cdot006006$, &c.

From this, it follows that the denominator of a circulating decimal consists of as many 9s as the decimal circulates figures; but any number of 9s is 1 less than the power of 10, whose exponent is equal to the number of 9s; and therefore, in estimating the values of circulating decimals, which are the numerators of fractions having denominators of this form, 9 in the right hand place must count as 10, because it counts as 10 in the denominator, in order to make it a power of 10. This principle, which is also explainable upon other grounds, must be attended to in calculations into which circulating decimals enter.

It is 1 in the last figure which the denominator of a circulating decimal requires to turn it into an exact power of 10; and as proportional parts must be added to both terms, in order not to alter the value of the fraction, the same part of 1 in the numerator must be added to that term, as the numerator is of the denominator. Thus the circulating decimal 19 is changed to a common decimal by affixing $\frac{19}{99}$, in this form $\cdot 19\frac{19}{99}$, which last expression, with the denominator written down, is $\frac{\cdot 19\frac{19}{99}}{1\cdot 00}$.

In the second place, every fraction whose denominator is 1 greater than any power of 10 circulates twice as many figures as the exponent of that power; 11 circulates two figures, 101 circulates four figures, 1001 six figures, and so on. If the numerator in these cases is 1, an 0 more than the exponent of the power of 10 must be annexed before the first figure of the quotient can be obtained; and that figure will be 9, so that the circle of the decimal will consist first of as many 0s, and then of as many 9s, as there are 1s in the exponent of that power of 10 which is 1 less than the denominator. Thus $\frac{1}{11} = \cdot 09$, &c.; $\frac{1}{101} = \cdot 0099$, &c.; $\frac{1}{1001} = \cdot 000999$, &c.; and so of all other denominators of that form. If these decimals are multiplied by the denominators it will be found that the products consist of as many 9s as there are figures in the circle; and hence again, in so far as they are concerned, we have a proof that the denominator consists of as many 9s as the decimal circulates figures.

We may arrive at the same conclusion directly, and more generally, thus:—

Let ·285714 be a circulating decimal, its denominator is as many 9s as there are figures in the circle, that is, 999999, or the fraction is $= \frac{285714}{999999}$.

Shifting the decimal point six figures to the right multiplies the decimal by 1000000. Hence,

$$\begin{aligned} 1000000 \text{ times is } &= 285714{\cdot}285714, \text{ \&c.} \\ \text{Subtract 1 time } &= \quad 0{\cdot}285714, \text{ \&c.} \\ \hline 999999 \text{ times is } &= 285714{\cdot} \end{aligned}$$

Therefore 1 time $= \frac{285714}{999999}$; or the denominator is as many 9s as the decimal circulates figures.

If the circle contains the maximum number of figures, that is, one less than the number of 1s in the denominator of the fraction, it is evident that those figures must follow one another in the same order of succession, whatever may be the numerator of the fraction. Only the decimals answering to different numerators will begin at different parts of the series. In such cases, if we get the decimal answering to numerator 1, we have all the others by examining what will arise from multiplying the first and second figures of this by the numerator. Thus $\frac{1}{7}$ is = ·142857; and as this contains the maximum number, or only one figure less than the 1s in 7, we have only to compare the product of ·14 by the numerator to see where we must begin. Numerator 2 will begin ·28, numerator 3 will begin ·42, numerator 4 will begin ·57, 5 will begin ·71, and 6 will begin ·85; and in each case the other figures will follow in their order.

When we know that a decimal contains the maximum number of figures in a circle we can get the corresponding fraction with less labour than that of reducing to the lowest terms, in the manner formerly explained. Thus,

From	10 times ·142857 =	1·428571, &c.
Subtract	3 times ·142857 =	·428571, &c.
Remains	7 times ·142857 =	1·
But if	7 times ·142857 =	1, then 1 time = $\frac{1}{7}$.

We shall next consider the decimals arising from those fractions, the denominators of which contain 2 or 5 once, or oftener, as factors, as well as other factors which are not divisors of 10. These will be best illustrated by a particular case, and we shall take $\frac{53}{56}$, the factors of the denominator of which are, $2 \times 2 \times 2 \times 7$. If we divide the numerator, 53, by these in their order, we shall obtain the decimal in the same manner as if we divided by 56 at once. $\frac{53}{2} = 26{\cdot}5$, $\frac{26{\cdot}5}{2} = 13{\cdot}25$, $\frac{13{\cdot}25}{2} = 6{\cdot}625$; so that if the given fraction is in its lowest terms, there are as many places of terminate decimals as the number of times that 2 is a factor; and it would evidently be the same in the case of 5. Let us now divide the last result by the remaining factor, 7, and we have, $\frac{6{\cdot}625}{7{\cdot}} = {\cdot}946\frac{3}{7}$, or, continuing the division, we have ·946428571$\frac{3}{7}$, which consists, first, of three figures of common or *terminate* decimals, answering to the three times that 2 is a factor of 56, and then of a circulating decimal of 6 figures, answering to 7, the remaining factor.

The first three figures are evidently $= \frac{946}{1000}$, because they are common, or terminate decimals; and the six figures of the circulating part are $\frac{428571}{999999000}$, because they are a circle of six figures, and begin at the fourth place from the decimal point.

The two parts of the decimal have not the same denominator, or even denominators which can be considered as standing in the same relation to the scale of numbers, for the denominator of $\frac{946}{1000}$ is a power of 10, namely, the third power; and the denominator of $\frac{428571}{999999000}$ is 1000×999999, or by $1000000-1$. But if we multiply both terms of $\frac{946}{1000}$ by $1000000-1$, we shall obtain a fraction equal in value with $\frac{946}{1000}$, and having the same denominator as the other; for $\frac{946 \times (1000000-1)}{1000 \times (1000000-1)} = \frac{946000000 - 946}{1000000000 - 1000} = \frac{945999054}{999999000}$, which is the value of the terminate figures, answering to the denominator of the circulating part; and adding them, we have the whole decimal $= \frac{945999054}{999999000} + \frac{428571}{999999000} = \frac{946427625}{999999000}$, the numerator of which is less than the *mixed* decimal by the terminate figures, that is, by 946.

Hence, in order to change a mixed decimal, or one containing terminate figures and then a circulating part, to a fraction, we have merely to subtract the terminate figures from the right of the whole, and the remainder will be the numerator, while the denominator will consist of as many 9s as there are circulating figures, with as many 0s after them as there are terminate ones.

In the *Arithmetic* of Interminate Decimals, it is necessary to attend to their denominators, in order to make the compensations which are necessary from those denominators not being complete powers of 10, or not being equal to each other; and as the making of these compensations is calculated to give a good deal of insight into the nature of numbers, and to teach expertness in the use of them, the time which may be devoted to it is usefully spent.

The first thing to be attended to is, to understand how they may be reduced to a common denominator, so that they may be added or subtracted with perfect accuracy.

If the circulating figures begin at the decimal point, they may be reduced to a common denominator by extending them to as many figures as there are in the least common multiple of the number in each. Thus, ·29, &c., ·314, &c., ·4231, &c., and ·142857, &c., will all have a common denominator, if extended to twelve places of figures; thus:—

·292929292929	29, &c.
·314314314314	31, &c.
·423142314231	42, &c.

The common denominator being twelve 9s, or 999999999999.

We also perceive that there will be 1 to carry from the next repetition of the circle, as marked a little to the right; and if what should be carried is attended to, the addition is the same as that of any other numbers. The sum of the above (carrying or adding 1 to that of the right-hand figures) is,

1·030385921575 0303, &c.,

which circulates the whole twelve figures. The sum, in these cases, never can circulate more than the number of figures in the extended circles; and it may circulate less, or become a terminate decimal, or an integer without any decimal; but there are no means of ascertaining this beforehand.

The subtraction is as simple as the addition, for we have only to see whether it is or is not necessary to add 1 to the right-hand figure of the number we subtract.

When the circulating parts do not begin at the decimal point, we must extend all the circulating parts till they are as long as the longest terminate one, and beyond that to a complete circle or multiple of all their numbers of figures.

This extending does not alter the value, for we may consider any number of figures on the left hand of a circulating decimal as terminate, without in the least affecting the value. Thus, if we mark the beginning and ending of the circulating part by an inverted comma, we have,

·'412857,' ·412'857412,' and ·41285'741285,'

all equal to each other; for if we subtract the figures which we mark as terminate from the right, we have the same number of 0s on the right both of the numerator and denominator; thus, ·'412857' $= \frac{412857}{999999}$, ·412'857412' $= \frac{412857000}{999999000}$, and ·41285'742185' $= \frac{41285700000}{99999900000}$; and these are all exactly the same fraction, if we leave out the 0s, which, being the same both in the numerator and the denominator, do not alter the value.

By combining the results of these investigations, we obtain this formula for the general addition of decimals, whatever they may be, terminate, circulating, or mixed:—extend the interminate ones till they contain as many figures as the longest terminate, and after that extend them to as many figures as are the least common multiple of the places in each circle; then see what would have to be carried from the addition of another circle, and, taking in that with the sum of the right-hand figures, add them as if they were integers.

In order to acquire readiness in the management of these decimals, it is best to take examples in fractions, to change them to decimals, and adding both the decimals and the fractions to see if the sums are the same, which is most easily done by multiplying the terms of the decimal by those of the fraction inverted; and if the products are equal, all the operations are right. We shall take an example:—

Let the sum of $\frac{5}{16}+\frac{9}{14}+\frac{49}{52}+\frac{101}{117}$ be required.

First by fractions:—find the least common multiple of the denominators.

	16,	14,	52,	117
÷ by 2,	8,	7,	26,	117
÷ by 2,	4,	7,	13,	117
÷ by 13,	4,	7,	1,	9

Then $4 \times 7 \times 9 \times 13 \times 2 \times 2 = 13104 =$ least common multiple of the denominators, or common denominator of the fractions.

To find the numerators answering to this denominator:—

$$\frac{5}{16} = \frac{\frac{5 \times 13104}{16}}{13104} = \frac{4095}{13104}, \text{ numerator } 4095$$

$$\frac{9}{14} = \frac{\frac{9 \times 13104}{14}}{13104} = \frac{8424}{13104}, \text{ numerator } 8424$$

$$\frac{49}{52} = \frac{\frac{49 \times 13104}{52}}{13104} = \frac{12348}{13104}, \text{ numerator } 12348$$

$$\frac{101}{117} = \frac{\frac{101 \times 13104}{117}}{13104} = \frac{11312}{13104}, \text{ numerator } 11312$$

Sum of the numerators $= 36179$

Sum of the fractions $= \frac{36179}{13104}$; or, dividing the numerator by the denominator $= 2\frac{9971}{13104}$; or, again, changing the fractional part to a decimal by adding 0s to the numerator, and dividing by the denominator till we find the quotient to circulate by the return of the same remainder, we have the sum = 2·7609'126984,' in which the inverted commas mark the circulating part, which may be repeated as often as we please.

In this, and in all other examples in which multiplications or divisions of numbers occur, we shall not, in general, set them down at length, but merely indicate them by the sign and the result. For this we have two very conclusive reasons: first, it saves room; and, secondly, any one who may be inclined to learn the subject from this work, has these as dissected exercises, in which he has something to do, and yet cannot easily do wrong.

We shall now treat the same fractions decimally; that is, we shall find the decimal corresponding to each fraction, and, having found the whole, add them together, and if the sum corresponds exactly with 2·7609'126984,' the decimal of the sum of the fractions, as found in the above operation, we may conclude that the operation by decimals is correct. In this we shall not write down the process of finding the decimals at length; but if the reader wishes to learn the subject from this book, it will be advisable for him to perform the operations, and, generally, to perform all operations in which he finds merely the data and the results stated. The decimals of the above fractions, simply stated as they arise from dividing the numerator of each, with 0s annexed, by the denominator, are as follow:—

$$\frac{5}{16} = \cdot 3125$$

$$\frac{9}{14} = \cdot 6\text{‘}428571\text{’}$$

$$\frac{49}{52} = \cdot 94\text{‘}230769\text{’}$$

$$\frac{101}{117} = \cdot\text{‘}863247\text{’}$$

We have next to state these decimals so that they have a common denominator; that is, a denominator consisting of as many 9s as there are figures in the least common multiple of the number in all the circles, and as many 0s as there are figures in the longest terminate part. All the circles contain six figures, therefore six is the number of 9s; and ·3125 is the longest terminate decimal, and contains four figures; so that we have only to begin the circulating part of each with the fifth figure, and extend each circle to six figures beyond that; and it is convenient to write a figure or two of each circle after this, and at a little distance apart, in order to see what has to be carried. The operation will stand thus:—

	·3125	
	·6428‘571428’	57
	·9423‘076923’	07
	·8632‘478632’	47
Sum, as before =	2·7609‘126984’	

The *multiplication* of interminate or circulating decimals is attended with a little more difficulty, and before we proceed to explain the principle upon which it depends, it may not be improper to mention that all interminate decimals to which the principles of arithmetic can properly be applied, so as to obtain results which are perfectly accurate, must be circulating ones; that is, decimals the values of which can be accurately expressed by fractions. We have already shown that the number of places which a decimal circulates can never be greater than the denominator of the corresponding fraction — 1; and therefore, though we cannot beforehand tell the number of places which any decimal may circulate, without an operation much more laborious than the actual finding of the decimal itself, and which, therefore, it would not be judicious to perform, we always know the limit which the number of places in a circle cannot exceed. But there are other numbers, or, properly speaking, other quantities or values which cannot be expressed exactly by any fractions whatever; and as, when these values or quantities are less than 1, they are fractions of which the denominators are indeterminate or unlimited, the decimals corresponding to them never can circulate, and therefore all that we can obtain is an approximation. Such are called *irrational* numbers; and before the invention of decimal arithmetic, the management of them was attended with a great deal of labour.

The *multiplication* of interminate decimals is not a matter of very great practical importance, because, in real calculations, we can always obtain an approximation much nearer the truth than we can come in any of the fractional divisions which are made use of in business. It is necessary, however, to understand the principle, and also to be able to perform the operation.

Now the fundamental principle is this:—we are to consider by how much the circulating part of the decimal is deficient of what it would be if it were a terminate decimal; and this is obviously as much added to the numerator as will enable 1 to be added to the denominator, without altering the relative proportion of the two, or the value of the fraction. Equal parts of the two terms are what is necessary to add; but the denominator is always as many 9s as there are places in the circle; and therefore the part added to the numerator will be the circle of figures divided by as many 9s as their own number.

For example, ·'27' is $\frac{27}{99}$, or, adding 1 to the denominator, and a proportional part to the numerator, it is $= \frac{27\frac{27}{99}}{100}$, the fraction in the numerator of which is not in its lowest terms, being divisible by 9; and thus the entire fraction is $= \frac{27\frac{3}{11}}{100} = \frac{300}{1100} = \frac{3}{11}$.

From this example it will readily be seen, that when the circle is a single figure, it is deficient by $\frac{1}{9}$th; when two figures, by $\frac{1}{99}$th; when three figures, by $\frac{1}{999}$th, and generally by one part the denominator of which is as many 9s as there are circulating figures. Hence again it follows that, if a circulating decimal is multiplied by any number, or used in multiplying any number, it will require the product to be increased by its 9th part if there is one circulating figure, by its 99th if there are two, and so on in the other cases.

Thus, if it were required to multiply ·'27' by any number, say by 1·875, that is, to find the product of 1·875 × ·'27', the true product would evidently be—

$$1{\cdot}875 \times {\cdot}\dot{2}\dot{7} + \frac{1{\cdot}875 \times {\cdot}\dot{2}\dot{7}}{99}.$$

Performing these multiplications and divisions we have,

1·875 × ·'27' = ·50625

$\frac{1·875 \times ·'27'}{99}$ = ·00511'36'

True product = ·511'36.'

On examining this operation, it will be seen that the figures in the second line are exactly the same as those in the product, only they are 100 times less in consequence of the two 0s between them and the decimal point; so that the product is that of the factors considered as terminate, and then divided by 99, and multiplied by 100, that is, every 99 is changed to 100; but 99 is changed to 100 by adding 1, and therefore we may obtain the true product without the trouble of dividing, by merely repeating the first-found line of the product, and setting it back from the point two figures each time, thus:—

Terminate product =	·50625
	50625
	50625
As before . . .	·511'36'

Therefore we have only to find the product as if both factors were terminate decimals, and repeat it, setting it as many places to the right, at each repetition, as there are places in the circle, and continue the repetition till we find a circle in the product, which, in the case of a single interminate factor, can never be greater than the number in that factor.

This method of dividing by 99, and multiplying by 100, or of changing 99s to 100s, may be used in cases where decimals are not concerned; and it may be extended to other cases, as, for example, to change 96s to 100s, or 24s to 25s. The advantage of this last case we shall see by and by.

When only one of the factors is interminate, it may, if the larger number, (that is, containing the most places of figures,) and especially if it has terminating figures before the circulating ones, be made the multiplicand, and the compensation may be effected by taking in with the multiplication of the right-hand figure of the circle whatever may require to be carried from another repetition of the circle, and then, when the lines come to be added, the circulating parts may be extended till they become similar, as in addition.

To illustrate this, let it be required to find the product of 4·23‘81’ by 2·5.

Multiplicand	=	4·23‘81’	81
Multiplier	=	2·5	
× by 2.	=	8·47‘63’	63
× by .5	=	2·11‘90’	90
Product	=	10·58‘54’	

When this method is adopted, it is best to begin with the left-hand figure of the multiplier, and set each line of the product a place farther to the right than the line before it, and it is easy to see what should be taken in in multiplying. Thus, in multiplying by 2 in this example, there is 1 to carry from the next repetition of the circle; in multiplying by 5, there is 4 to carry; in adding, there is 1 to carry; and it so happens that the circle begins at the third figure, as in the multiplicand.

We have next to consider the case in which both factors are interminate decimals, and it will save time if we at once take the most complicated part of this case, that in which each factor contains both terminate figures and circulating ones; and it is obviously of no consequence whether the terminate figures are wholly integers, wholly decimals, or partly the one, and partly the other. In order to see the general principle in this case more clearly, it would be better to take it algebraically, before we proceed to an example in numbers.

Let, then, $a+b$ be one mixed factor, of which a represents the terminate figures, and b the circulating ones; and let $c+d$ be another mixed factor, in which c is terminate, and d a circle; the product is,

$$(a+b) \times (c+d).$$

The multiplication, performed as already explained, stands thus:—

$$\begin{array}{c} a + b \\ c + d \\ \hline ac + bc + ad + bd. \end{array}$$

As all the four terms of this product consist of different combinations of letters, no two of them can be added together; and when we examine them according to the conditions, they are as follows:—

$a\,c$ = product of the two terminate parts.
$b\,c$ = product of the terminate part of the second by the interminate part of the first.
$a\,d$ = product of the terminate part of the first by the interminate part of the second.
$b\,d$ = product of the two interminate parts.

The sum of these four products make up the entire product of the two next decimals: the first of them is found as in common numbers; the second and third are cases of one terminate and one interminate factor; so that the fourth, or the two interminate factors, is the only one which remains to be noticed. In practice, as we said, this is seldom useful, as it is better to find the corresponding fractions, or to multiply the factors by any numbers that will clear away the interminate parts, and divide the product, when found, by those numbers; but still, as the case may occur, we shall very briefly state the formula for its solution.

If a is the figures in one of the circles, and b the figures in the other; also, if n is the number of those figures in the first, or, which is the same thing, the number of 9s in the denominator, and m the number of figures and number of 9s in the denominator of the second, the completed fractions will be, the first $= a + \frac{a}{n}$, and the second $= b + \frac{b}{m}$; and the true product will be that of $a + \frac{a}{n}$, by $b + \frac{b}{m}$, which, found as before explained, is $= a\,b + \frac{a\,b}{n} + \frac{a\,b}{m} + \frac{a^2\,b^2}{m\;n}$; of which it is unnecessary to give an example.

When interminate decimals occur in cases of division, the quotient may be obtained by the application of those principles in multiplication and subtraction which have been already explained; but, in general, every practical purpose is answered by taking an approximation, that is, simply continuing the division to some length when only the dividend is interminate, and making allowance in the multiplication and subtraction when the divisor is interminate. It is better, however, to try for some number which will exterminate the circulating part, in the case of an interminate divisor.

In those calculations in common matters of business where the general introduction of decimals would be of so much advantage, circulating decimals do not very frequently occur, neither are those which do occur of very formidable nature. In the denominations of British money, all the factors by which pounds are changed to farthings, are divisors of the sixth power of 10, that is, of 1000000, with the exception of 3; so that no decimal of a pound, expressible in farthings, can circulate more than a single figure; and every such decimal must terminate or circulate at the sixth figure from the point, if it does not do so previously. This is extremely convenient, and sufficient of itself to induce every one who has occasion to calculate prices, to make use of decimal arithmetic; the more so that, as we shall have occasion to show, the decimals of odd parts of a pound are just as easily written down as the parts themselves, and as there is not more difficulty in writing down the odd money which answers to the decimals.

In Troy Weight, the only factors which produce circulating decimals are two 3s; one in 24 grains, and the other in 12 ounces; and these produce 9, which is a circle of not more than one figure. In common Avoirdupois Weight, the only interminate factor which occurs is 7; and if all the statutory weights and measures which are used in Britain are examined in a similar manner, it will be found that the number of interminate factors, in their several denominations, is fewer than could have been expected, in a case where there was certainly no reference to decimals when the numbers were originally fixed on. We shall close this section by a short account of the method of turning the subdivisions of a pound sterling to decimals, or in finding the parts of a pound answering to any given decimal.

In the first place, as 20 shillings = 1 pound, 2 shillings make 1 tenth, and half any even number of shillings make tenths; so that we can always determine the first figure of a decimal with no trouble. If there is a shilling over the even number which make tenths, and also pence and farthings, these cannot, in any case, amount to tenths, but must be disposed of in the following places of the decimal.

Now here again the matter is very easy. There are 960 farthings in a pound, which wants 40 of being 1000; so that thousandth parts, that is, decimals to three places, give us the nearest farthing. But 40 is 1 twenty-fourth of 960; that is, the farthings in a whole pound want 1 twenty-fourth of their number of being 1000; and therefore, if 1 twenty-fourth of their number be added to any number of farthings, they will become 1000th parts of a pound, or, which is the same thing, decimals to three places of figures. Adding 1 for every 24 is, in round numbers, adding 1 twenty-fourth; and therefore we have only to turn the money which is over even shillings into farthings, and add 1 for every 24, to get the second and third figures of the decimal; and all the operations are of so simple a nature, that we can perform them as fast as the figures can be written. Thus, to turn 15*s.* 11¼*d.* to decimals of a pound: half the shillings is 7, and 1*s.* over; and 1*s.* 11¼*d.* is 93 farthings, and 93 contains three 24s, so that we must add 3, and a whole decimal is £·796. It must be recollected that the figures thus found must make two places, and consequently, if they amount only to one, the second place must have 0.

But it is not 1 for every 24 which we should add; it is 1 twenty-fourth; and here again we can have a modification of the way in which the compensations are made in circulating decimals. 4 times 24 make 96, and 4 times 25 make 100; therefore, if we take our number of farthings without any addition, and repeat them, setting the first repetition under the thousandths, and every succeeding one two places farther back, we shall get the exact decimal.

£0 15 $11\frac{1}{4}$	=	·793, as before.
93 × 4, and set back	=	372
372 × 4, and set back	=	1488
Again	=	59
Yet again	=	3
Exact decimal . . .		·7968750

This operation shows the principle, but it is a little clumsy, and may be simplified and performed mentally, by adding 1 for every 24, and rejecting the hundreds of the next step. Thus, to revert to our example, half 15 is 7; 93 farthings and 3 added are ·796; 4 times 96 = 384; blot out the hundreds, 3; add 1 to the units and tens for every 24, which is 3; and we have ·79687; multiplying 87 by 4 is 348; blot out 3; add 1 to 48 for every 24, is 2, which makes 50, or, for the whole decimal, ·7968750, as before. If the decimal terminates, it always does so in 25, 50, or 75, because these numbers multiplied by 4 leave no more units and tens to be added; and when it is interminate, it of course circulates a single figure.

The converse, or method of finding the shillings and pence to the nearest farthing, is still easier. ·05 is $\frac{1}{20}$th, and therefore the decimal of a shilling; wherefore, divide the first and second figures from the point by 5 for shillings; take 1 from every 25 of the remainder of the second figure, with the third after it, and the rest is farthings, which divide by 24 for pence.

Thus, £·796875 = 15*s.* $11\frac{1}{4}$*d.*

These methods are exceedingly simple, and it is well worth the hour's labour which it requires to learn dexterity and accuracy in the practice of them; but we can only recommend that, and pass to another section and branch of the science.

SECTION IX.

PRELIMINARY NOTIONS OF GEOMETRY.

As there are some of those considerations in the general science of quantity, which come next to be examined in that order which we conceive to be most conducive to clear and simple views of the elements of the whole, which have as much reference to geometry as either to arithmetic or to quantity generally, it will be proper to take some notice of the simpler portion of that branch, or rather application, of the general science.

It has been already mentioned, that geometry is the science of magnitude, or quantity considered as extended in some way or other, so as to occupy space; but that which occupies space is not the only subject of geometrical investigation, for, before we can arrive at a distinct notion of even the simplest body which can be supposed to occupy space, there are many elements to be considered, and many relations to be understood; for, though we do not, in our geometrical inquiries, trouble ourselves about those physical qualities of real bodies which form the distinctions between one kind of body and another, and which give to each of them its peculiar practical value, yet that which we call a body, or solid, is not an original perception of our senses, but a compound result of various relations of elements, all of which we must thoroughly understand before we can have a proper geometrical notion of the body or solid itself; and if any of those relations happen to be indeterminate, our knowledge of the solid is vague and imperfect. Thus, for instance, though we have a general notion that the earth which we inhabit approaches pretty nearly in form to a round ball, or globe, as we call it; that the diameter, or measure straight through the centre of it, is nearly 8,000 miles; that the diameter measured at the equator, or that region where day and night are about equal at all times of the year, is about 30 miles longer than the cross diameter extending from pole to pole, or from the one or the other of those places where we know, by well-founded inference, though not by positive observation, that the year consists of one day and one night, of nearly, but not exactly, equal length;—though we know all this, and though some of the most able men of all ages, having the best claim to be considered scientific, have devoted their best attention to the determining of the earth's figure, yet there are many particulars which prevent the results of their labours from being perfectly accurate, and even deprive them of the means of ascertaining the degree of accuracy which they actually attain. Nor is this the case with the globe itself merely, but with every portion of it, and with every portion of matter to which we practically apply, or can apply our geometry. If a field of ground, or even the length of a road, or the breadth of a river, or the distance of any one point of the earth's surface from any other point, is measured with the greatest care by two different surveyors, or even twice over by the same surveyor, the chance, nay, almost the certainty is, that the results will not correspond, and one would be half inclined to suppose that things alter their shapes and sizes for the very purpose of perplexing us in our measurements. Even the standard of our mensuration is liable to vary: if it is a piece of tape, or a twisted cord, it is shorter when the weather is damp than when it is dry; and if it is a metal chain or rod, it is longer when the weather is warm than when it is cold.

There are many other known causes of incorrectness than these, and of what are not known, but may exist, we are of course ignorant. Those circumstances are mentioned, not with a view of decrying the merits of geometry, but merely to show that, however any of our sciences may profess to be perfect in theory, we must not rest satisfied with a single branch, when we come to apply them to real practice, but must know the properties of things themselves, and the variations to which they are liable, as well as those abstract sciences of which we make them the subjects.

It will be readily admitted, that, as we have so many causes of error to contend with in the application of our geometry, it behoves us to be very accurate in that geometry itself, and not to allow ourselves to add the disadvantage of an imperfect and badly understood tool, to that of ungainly or unmanageable materials.

Many of the words which we use in a strict sense in geometry, are used much more loosely in common language, and therefore we require to make ourselves well acquainted with the difference between the scientific and the popular meaning. Indeed it is the want of attention to these differences, and the consequent looseness of our fundamental definitions in science, that the greater number of the hardships which we feel in the study of it, and the blunders which we make in its application, are mainly to be attributed.

Even the word *solid* has popular meanings different from its geometrical one; and the geometrical solid is not necessarily a solid body, or a body which has real existence at all; it is a certain portion of space, the boundaries of which, in all their dimensions and relations to each other, are known and determined; and if those dimensions and their relations to each other are the same, it is of no consequence whether the space which they bound is filled with any one kind of matter, solid, or liquid, or air, or whether it is empty space, or even has any existence. Its existence, full or empty, must, however, be possible upon geometrical principles; that is, there must be nothing absurd in any of the relations which the dimensions or other characters of the geometrical solid bear to each other; as, for instance, the solid must have length, and breadth, and thickness, that is, three dimensions situated with regard to each other, in directions afterwards to be explained; but whether any of these are the same with each other, or whether any one of them is the same at two points of the solid, must depend upon circumstances. So also the solid, in order to be geometrical, must have boundaries which inclose it everywhere, but do nothing more, and which, from their known figures, dimensions, and situations with regard to each other, determine the form of the solid.

We can determine nothing geometrically without measuring, and measuring by the application of a standard, which, as must be the case with all standards, must be of the same kind with that which we can measure; and not only must we have a standard and apply it, but we must begin at some beginning, and this beginning is the primary and simplest of all geometrical considerations. In order that we may have a definite or known measurement, we must have an end as well as a beginning; and as any distance or extent in space is the same, whether we measure it in one direction or in the opposite, the end and the beginning stand precisely in the same relation to the measure of which they are the terminations. Thus, for example, it would take exactly the same number of revolutions of the same carriage-wheel to pass over the road between London and York, whether the carriage began the journey at London and ended at York, or began it at York and ended it at London.

This simplest of geometrical elements is called a POINT, and, simple as it is, it is necessary to have very clear notions of it, because even it has given rise to some idle use of words. A point means position merely, and therefore it is not considered as occupying any portion of that space in which it is situated; and when we consider nothing but a mere point and space, we may regard any point, wherever we imagine it to be situated as the centre of space, that is, as being in the very middle of space, or having equal measures of extension in all directions around it. Thus, in whatever part of its orbit the earth may be, or on whatever part of its surface a spectator may be situated, the centre of his eye, while he surveys the heavens around him, is to him the centre of space. But when a point has reference to any particular dimension or measure, it may be either of the extremities of that measure, or anywhere between them, to which allusion may have occasion to be made; yet, having no dimensions itself, it can form no part of any dimension; it bears, in fact, to extension nearly the same relation that 0 bears to number in arithmetic, for as no multiplication of 0 can produce even the smallest possible number, so no repetition of a point can produce even the smallest dimension; and so also, as no division by 0 can in the slightest degree diminish the smallest number, no division by points can diminish the smallest possible dimension.

A LINE is the geometrical element next in simplicity. There are various kinds of lines, straight lines and crooked or curved lines, and the latter may have single or double curvatures: thus, a road which makes a sweep upon perfectly level ground, is a familiar instance of a curve of single curvature, and a road which makes a sweep across a hill or a valley, is a familiar instance of a curve of double curvature. A curve of single curvature may either curve to one hand only, as is the case with the outline of the full moon, or it may curve first to the one hand and then to the other, as a serpent does when it crawls on level ground, or an eel when it swims in the water; and then it is a curve of contrary flexure, or it may be curved round a central line, like the thread of a screw. In fact, there is an endless variety of curved lines, or *curves*, as they are usually called, but they do not properly belong to the mere elements of geometry.

A STRAIGHT LINE is the simple elementary line, and it is this which we always mean when we use the word line in a geometrical sense, without using some other word to express the kind of line.

A straight line is the shortest distance between those points which form its extremities, and therefore, as there cannot be two shortest in the same case, there cannot be two straight lines joining the same two points, neither can any portion of space be inclosed by, or situated between, two straight lines which meet each other at any two points.

As a straight line is extension in one direction only, it can have no material existence as a separate quantity, but merely means the shortest distance in some direction.

If the points which mark the extremities of a straight line are known, the magnitude or length of that line is determinate; and if these points, or any other two points in the line have fixed positions, the position of the whole line is also determinate. But, because a straight line has no material existence, we can imagine one to be drawn from any point, in any direction, and to any distance; or we may imagine a straight line to be continued to any additional length. We may also imagine a straight line to be drawn through any point, in any direction, and to any length both ways; or we may also imagine an endless number of straight lines to be drawn through the same point, all in different directions.

A SURFACE, OR SUPERFICIES, is the next element in the order of procedure from the more simple to the more complex, and of this there are, as there are of lines, several kinds, independently of the outline or *figure* of the boundary. A plane surface, or simply a PLANE, is the elementary one, and it is that which, in common language, we are accustomed to call "perfectly level," or "perfectly flat." The usual definition is, that, if a straight line touches a plane in two points, it must touch it in every point, so far as both extend; and if two straight lines are drawn across each other, a plane will touch all the four crosses, or branches formed by the crossing, as far as their arms and the plane extend: also, if two planes touch each other in three points, they will touch each other in every point, as far as they extend.

A plane surface which has definite boundaries is called a figure, and those boundaries must be either straight lines or curves of single curvature. If the boundaries are straight lines, the figure is called "right lined," or *rectilineal*, the word "right" being, in geometry, often used as synonymous with the word straight.

A surface, whether plane or not, has no separate material existence, any more than a point or a line: it is the boundary of a solid, whether considered as mere space, or applied to a portion of matter; but it is not a part of the solid in either case; for as no multiplication of a point could produce a line, no multiplication of a line can produce a surface, and no multiplication of a surface can produce a solid. But lines and surfaces are *magnitudes*, or geometrical quantities, as well as solids, for they can extend a greater or a less distance in space, only a line extends in one direction in space, a surface in two directions, and a solid in three.

Arithmetically, a line is represented by a simple number, a surface by a product of two factors, and a solid by a product of three factors. *Length* and breadth are the factors of a surface; *length*, *breadth*, and *thickness* the factors of a solid; and both breadth and thickness in a solid "*stand* in the same relation" to length that breadth does to length in a surface.

As straight lines have only one quality, namely, length, they can *bear* only one kind of relation to each other, that of being of the same length, or different lengths; and therefore it is important to consider in what relation they can, in addition to this, *stand* to each other. Now the only circumstance in which a straight line can vary, or in which one straight line can be different from another, except length, is *direction;* and the distinction here, as in the former case, is, that any two lines must either stand in the same direction with each other, or they must stand in different directions.

This relation of lines, in respect of the positions or directions in which they stand to each other, is one of the most important considerations in geometry; but it is one in which beginners usually feel some difficulty, and, indeed, there is at least one point connected with it, which, though clear enough, cannot be explained or proved in a rigidly geometrical way, and therefore it is a portion of the subject to which the most careful attention should be paid.

There is, indeed, another difficulty; for when we speak of the relation in which two lines stand to each other in regard to position or direction, we speak of a quantity which has no connection with the length of the lines as magnitudes, and therefore that which we seek to know hardly admits of ordinary description. Some general notion of the two leading divisions of this relation, having the same direction, and having different directions, may, however, prepare us for understanding more readily the particular cases.

What is meant by two straight lines having *the same direction?* This is the part of the case in which technical geometry has usually broken down, in the hands even of the ablest of its professors, and therefore it is very trying for popular illustration; but still we must attempt it, and if we fail, we shall fail only where many have failed.

Let us analyse it, and inquire, first, what is meant by the direction of a line, and then when the directions of two lines can be said to be the same. In the case of a line, the direction is the stretch of the line, the way that it extends, or goes out or away from any point; and hence the Greek geometer defines a straight line as that which goes, *εξ ισου*, (*equaliter*, "goes equally"—*ex æquo*, "equally from") "evenly out," without turning or deviation in one way or another; a line, in short, which is all in one direction. This direction is one and the same through the whole of one straight line, through all parts of it, and through every continuation of it which can be made either way through space. But the direction is not fixed or definite with regard to anything but the line itself, unless the line passes through two fixed points, and then it is in the direction of those two points. Thus, if we suppose one point at London, and another at York, a straight line passing, or which would pass through both these points, has the direction of them, whether we suppose it to extend all the way, from point to point, and no farther, or imagine it continued through space from the one point or from both, or whether we consider any small portion of it, as, for example, a foot, or an inch. It is evident that there can be only one straight line passing through any two points; and as the parts of this line, in what manner, or in what lengths soever they may be divided, have all, from the very definition, and even the possible notion of the line itself, one direction, the parts of this one line (that is, of any one straight line whatever,) do not form those lines which we say have the same direction, because to say that any quantity, whether line or anything else, is the same as itself is a truism, or identical proposition, that is, the mere repetition of the same truth in different words, which, of course, conveys no information whatever. Hence we arrive at one negative part of the notion of straight lines having the same direction; that is, they cannot both pass through the same two points; but as this is a property common to all straight lines it gives us but little information.

Every straight line must lie wholly in one plane, from the very nature of a straight line and a plane; and it follows from this, that two straight lines which have the same direction must be in the same plane, and this brings us a little nearer to the definition, and would bring us to it altogether if geometricians would acknowledge their obligations to a principle which they reject in words, but without which they cannot advance one step in reality.

This principle is *motion*, or progress through space, as from one point to another; and it is from this motion, whether we admit it or not, that we derive both our original conception and our geometrical representation of a line. Our original conception of extension is progressive motion; when we look at any object we judge of the measure along or across it by carrying the eye, or making the axis of vision travel from the one extremity to the other; when we measure any line in practice we pass over it with the measuring instrument; and we have no means of obtaining a knowledge of the length of any one line whatever, unless we are enabled to connect it, by clear and well-understood relations, with some line the length of which we have discovered by passing successively over the different parts of it, in some way or other. When a ship traverses the ocean, a bird flies through the air, or the earth or any other celestial body careers on through the regions of space, though neither of these leaves any trace of its progress behind it, we are in the constant habit of saying that each "describes a line" as its path, and we speak of the course of the ship, and the orbit of the revolving planet, as if they were actually existing quantities, and trace them upon our maps, or other tablets of instruction and reference, in the same manner as though there were lines answering to them, palpably and permanently drawn in nature, along which we could proceed from the one extremity to the other, step by step, as we do along a common highway upon the land; in fact, if we are to know any line originally, and not by conclusions deduced by reasoning from another line or lines, we must travel along; and if the original demand on us were the length of the line from London to York, we have no original means of satisfying the demand without travelling from London to York, or from York to London, and taking heed of our progress as we went. View the matter as we will, our notion of the distance, extent, or line between one point and another, always resolves itself into motion from the one end to the other; and therefore, instead of motion being an ungeometrical conception, it is really the primary foundation of all geometry, and without it we could have no knowledge beyond that of an unextended point, if indeed we could possibly have so much.

And when we come to get a representation of a geometrical line, by what means do we obtain it but by taking the pen, the pencil, or the other drawing instrument, and *moving* it from the one extremity to the other? and until this motion has been performed there can be no representation of the line. Even they who reject motion as ungeometrical never can perform a single problem in their science, or explain the performance of it, without *drawing* lines, and there is no way of drawing a line but by giving that which is drawn a *motion* from the one extremity of the line to the other. Upon no account, therefore, ought the idea of motion to be excluded from the very first elements of geometry; for it is really the tie which binds the truths of geometry, both to the material world and to the human understanding, and it were as hopeless to deprive man of the use of all his senses in order that he might thereby better study the productions of nature and the works of art, as to exclude the doctrine of motion, in order that geometry might be more successfully studied.

We have said that lines which are in the same direction must be in the same plane, and that they cannot meet in two points, however near or however far asunder these points may be, without coinciding altogether, or being one and the same. Now straight lines which are in the same plane must either, if continued far enough, meet each other in one point, or they must not; and therefore it may be said of any two straight lines in the same plane, that they must both be directed or tend to the same point, or they must not; and if they both tend to the same point, or, which is the same thing, lie in the direction of the same point, they must reach that point if continued far enough; and they must reach it as straight lines, that is, without the slightest change in the direction of either; and farther, as there is no reason why we should suppose them to stop at this point, they must cross one another at the point where they meet, and change sides, so that the one which was formerly toward the left hand shall now be toward the right.

Let us take an illustration :—if we stand anywhere on the road between London and Bath, and imagine one straight line to proceed from London, and another from Bath, each in the direction to York, the Bath line if we are looking toward York will be on our left hand, and the London line on our right, and the distance between the extremities whence they proceed will be the distance between London and Bath.

Now let the extremities of the lines, or which is the same thing, that which we imagine to describe, each (a carrier pigeon if you will) speed onward, by motion, to the point to which both are directed, namely, to York, is it not evident that they will come nearer and nearer to each other as they approach nearer to York ; and that, when they pass that city, the Bath pigeon will carry onward his line to the right and the London one to the left? Moreover, as they get farther and farther beyond York they will get more and more apart from each other, and never by any possibility again be both directed toward the same point. In this instance it must be supposed that the pigeons move in perfectly straight lines; and it is of no difference with regard to the position or direction of the lines, whether they both arrive at York at the same time or the one before the other.

The lines which we have instanced as meeting at York, are *inclined toward* each other in the direction toward that point, whether we consider them on the London side of York or on the opposite side; and if we consider them as viewed from York, the point where they cross each other, they are *inclined from* each other. We are considering them both as straight lines, and therefore their inclination must be the same to whatever length we may suppose them to be continued either way. Their positions with regard to each other are reversed on opposite sides of the point where they meet and cross each other; but still, as straight lines always preserve the same direction (which is in fact the very meaning of straightness), their inclination must be everywhere the same in amount. If we view them as from the point where they cross, they must have the same inclination from each other on the one side of this point and on the other; and if we view them toward the crossing point, they must have the same inclination toward each other when viewed either way, and whether their lengths be greater or less, whether not the thousandth part of an inch or greater than any distance to which we can give a name.

As this is exactly the fact of the case in which systematic geometers find, or rather make the difficulty, we have taken an illustration from two particular lines, in order to impress the principle more strongly upon the reader; but it will be easily understood, from what we have said of the two straight lines beginning at London and Bath, meeting and crossing each other at York, and then proceeding onward, at the very same inclination as before, but in reversed positions, that the fact must be true of any two straight lines whatever which are situated in the same plane. We may therefore assume it as a necessary conclusion, from the very nature of straight lines, from the fact that every straight line lies wholly in one unchangeable direction to whatever length it may be continued, that if any two straight lines situated in the same plane have any inclination whatever toward each other at the one extremity, they must continue to have the same inclination towards each other, how far soever we continue them in that direction; and that, be the inclination however so small, they must meet if we continue them far enough. Also, if this be the fact, as it evidently is in the case of their actual continuance, we may assume it as a fact in the case of any two lines, however short, which have any inclination towards each other.

And if they are inclined towards each other in the direction of their extremities one way, they must be exactly as much inclined from each other in the direction of their extremities in the opposite way; and, instead of meeting and crossing each other in this opposite way, they must get farther and farther apart from each other, the farther they are continued.

That the inclination must be constant, that is, the same at every point of the lines, follows also from the very fact of their being straight, because the inclination could vary only in consequence of a change in the direction of one of the lines, or of both, and if this change took place one or both would cease to be straight. Therefore, we may repeat again as a truth as legitimate and as clear as any in geometry, that "if two straight lines situated in the same plane have towards each other any inclination whatever, they must, if continued far enough, meet each other in the direction towards which they are inclined."

But it may be desirable to examine this a little farther: the lines, whatever may be their length and inclination, must be at some definite distance from each other at those extremities which are farthest apart, and they must be at a less distance than this at those extremities which are nearest to each other, otherwise they could not be inclined to each other. Now, whatever these distances may be individually, as the one is, in virtue of the very fact of the lines being inclined greater than the other, they must have a difference, and this difference must be the same in equal lengths of the lines, otherwise there would be a shifting in the direction of one or both of them, and consequently they would cease to be straight.

Here, again, it may be desirable to refer to a particular instance of definite lines, for the sake of illustrating the general argument:—well, suppose the lines are a foot in length each;

that they are 10 inches apart at the one extremity, and 9 inches at the other, the difference between their distance is 1 inch. Continue them another foot in the direction in which they are inclined to each other, and their distance will diminish another inch, or their extremities will be 8 inches apart, continue them a second foot and they will be 7 inches apart, a third and they will be 6, a fourth and they will be 5, a fifth and they will be 4, a sixth and they will be 3, a seventh and they will be 2, an eighth and they will be 1, a ninth and they will be 0, that is, they will meet each other. Continue them beyond this, and they will open an inch from each other for every foot they are continued, only the positions will be reversed in respect of what they were before the point of meeting was arrived at.

Now, though the definite measures which we have assumed for the length of these lines, and for their distances at their opposite extremities, determine the point at which these particular lines meet, which, by looking back to the analysis, will be found to be 10 feet from the most distant, and 9 from the nearest extremities of the lines; yet, if we take general expressions for the lengths and distances, we shall obtain a general expression in terms of these, for the point at which the lines must meet.

Thus, let $a =$ the length of the lines, d the difference of their distances at the two extremities, md the greater distance, and $md - d$ the less, which expressions are merely the former ones generalised. Now, when the lines are extended to $2a$ their distance will be $md - 2d$, at $3a$ it will be $md - 3d$, and so on, till at ma from the most distant extremities of the lines, it will be $md - md = 0$, that is, the lines will meet.

Wherefore, it is not only universally true that two straight lines which are in the same plane, and have an inclination to each other must meet if produced far enough; but also, that if the length of the lines and their distances from each other at both extremities are given, we have data sufficient for calculating by a very simple operation the length to which they will require to be continued in order to meet; and their length. We have seen that if d be the difference of distances for any length a, and md the greater distance, then md becomes $= 0$ when the length of the lines reckoned from their most distant extremities becomes $= ma$, therefore the difference of the distances is the same part of the greater distance that the given length of the lines is of the length to which, including the given length, they must be extended in order to meet.

To whatever extent two straight lines in the same plane are inclined toward each other at their extremities one way, and from each other at their extremities the other way, their directions differ from each other by exactly this quantity of inclination, whatever it may be; and consequently, on the other hand, if two straight lines in the same plane have no inclination toward each other one way, and from each other the other way, these two lines have exactly the same direction, that is, whatever can be proved by the direction of one of them will necessarily hold good as equally true of the direction of the other. Nor is this confined to two lines, for we may state generally that if any number of lines whatever in the same plane have no inclination to and from each other, they are all in the same direction, and whatever can be proved of the direction of any one of them, will follow as necessarily true of that of each and all of the others.

Thus we have two distinct classes of straight lines in the same plane; first, lines which are inclined to and from each other, which lie in directions differing from each other by the exact measure of their inclination, and which, if produced far enough, must meet the length required for this purpose, also varying with the inclination; secondly, lines which have no inclination to each other, but are in the same direction, and which, consequently, have not the property which depends upon inclination, namely, that of meeting each other when produced. Lines which have no inclination are called PARALLEL LINES, which means that they are "spoken of as away or apart from each other," which name comes very nearly to the definition, as it involves the notion that, produce them ever so far, there is no point at which we can speak of them as being together: and this differs but little from Euclid's definition, which, though it has not been adopted by all geometers, is at once the simplest and the best:—"Straight lines which are in the same plane, and being produced ever so far both ways do not meet, are called parallel lines."

It will be perceived that the words "being produced ever so far both ways" in this definition take for granted that very principle of motion which the more rigid geometers profess to reject; and not only motion, but motion "both ways," that is, from both extremities of the lines, and continued indefinitely both ways "ever so far." Now if motion is assumed as a *postulate* of action, the possibility of performing which is self-evident; and farther, if the same notion is involved in the three postulates of, joining two points, producing a straight line, and drawing a circle, it is not easy to see upon what principle it can be excluded in the definition of straight lines in the same plane which are inclined to each other; and that this should be the case is the more wonderful, and the more to be regretted, that its admission there removes much of the labour and more of the difficulty of elementary geometry.

It may be said that the words "inclined to and from each other" stand in need of explanation; but the word "inclined" is as generally understood as any word in language; and if we were to give an elaborate definition of every word we make use of in our attempts to explain the principles of science, we should never be able to let anybody see one of those principles through such a misty multitude of words as would in that case envelope them all.

We may now alter the expression a little, and say, "all straight lines in the same plane which are not parallel to each other, must, if produced far enough both ways, meet each other either the one way or the other, but not both."

They must meet when produced at those extremities where they are inclined *to* each other; and they must become farther apart the farther they are produced at the opposite extremities, or at those at which they are inclined *from* each other; but throughout the whole of their length, whatever it may be, and whether they are produced till they meet or not, their inclination to and from each other is exactly the same, and cannot vary unless one or both change in direction, and thus lose the character of straight lines. We have used the words "from" and "to" together in the course of these explanations, because they are inseparable by the very nature of the case; an inclination *to* the one way being as necessarily an inclination *from* the other way, as a road which when taken at the one end leads directly to London, leads as directly from London if we take it at the other end.

The inclination of lines to each other furnishes us with another set or kind of quantities in elementary geometry besides *lines*, *surfaces*, and *solids*. The measure of this inclination is called an ANGLE, which means a corner; and as we have spoken of *straight* lines, which are also called *right* lines, in the same *plane*, the simplest angle which occurs in elementary geometry is—

A PLANE RECTILINEAL ANGLE, which may be defined as "the corner, or opening made at the point where they meet by two straight lines which are inclined to each other."

As the inclination and the angle which measures it are quantities, they must, like other quantities, admit of variation, that is, there may be an endless variety of angles answering to and measuring an endless variety of inclinations, just as there may be an endless variety of lengths of lines, extent of surfaces, and capacity or content of solids; but as the angle depends altogether upon the inclination of the lines, and not upon the length of any one of them or of them both, or even on the direction in which any one of them extends, an angle is not a quantity of the same kind with a line, neither is it a quantity which can be accurately expressed by any product of straight lines, as a surface is by the product of length and breadth, and a solid by the product of length, breadth, and thickness.

The four different kinds of quantities, lines, angles, surfaces, and solids (for that is the proper order in which to take them), of which we have attempted to give as clear and simple an explanation as possible, are the only quantities which are purely geometrical; and with the exception of one curved line (to be afterwards described), and the curved surfaces and solids which are founded on the line, the relations of straight lines, rectilineal angles, plane surfaces, and plane solids, include all which belongs to elementary geometry.

We have endeavoured to explain the nature of the quantities at greater length than is usually done, and in different terms, though not upon different principles; for we trust it has been shown that motion, which we have taken as an element, has been tacitly assumed by every geometer; and we feel convinced that it has been this latent principle, which works powerfully, but unconfessedly, which has made the science of geometry—a science beautifully clear in itself—so perplexing to the majority of students.

SECTION X.

GEOMETRICAL QUANTITIES, METHODS OF EXPRESSION, AND DEFINITIONS.

In the general account of the elementary notions of geometry and geometrical quantities given in the last section, we studiously avoided all allusion to the methods of expressing quantities geometrically, to the short elementary definitions, self-evident principles (or *axioms*), and also to the objects of geometrical operations or inquiries, as usually given in books on the elements. When the general notion is explained by allusion to a particular symbol, there is some danger that the symbol will lay hold of the student's conception, and particularise it; and this is especially the case when the quantity under explanation is a relation, and as such, not expressible by any separate symbol, but merely by the position of those other quantities of which it is a relation. Thus, for instance, a line, a surface, or a solid may be represented by a picture, or *diagram*, as it is usually called; but no diagram can represent simply and singly that which we mean by an angle. This magnitude (an angle) can be represented in a diagram only by the two lines of whose inclination it is the measure; and as these lines must, in any diagram which can be drawn, have some visible length, and also include between them, as far as they extend from the point of meeting, some visible portion of surface, it is very difficult for a beginner to avoid mixing up the notion of the lengths of the lines, and also that of the space between them, with the proper notion of the angle. It will be found that, in consequence of this confusion, those who have made but little progress in geometry, even though they have made some, have very vague and confused notions of what is really meant by an angle, independently of the lengths of the lines of whose inclination it is the measure, and of the quantity of surface which may be contained within or between those lines. If the explanation which we gave in the last section has been read and studied with due attention, the reader will not find much difficulty in forming a correct notion of what an angle is, without mixing it up with any notion of the length of particular lines, or the extent of particular surfaces; and if we have succeeded in doing this, the reader will have gained more than he is aware of.

We may now, therefore, proceed to point out the modes by which quantities are geometrically represented, and the names which are given to a few of the simpler modifications of them; and in the mean time we shall confine ourselves to the elements of Plane Geometry, that is, to lines, plane rectilineal angles, and surfaces; only, because the knowledge of the only curve which enters into the elements of plane geometry is necessary, in order rightly to understand the distinctions of those leading varieties of angles which we require to define at the outset, we shall include that curve among lines, though it cannot appear as a line without appearing at the same time as the boundary of a surface or figure.

1. Of Lines.

There are only two kinds of line in elementary geometry, the straight line and the circle.

Straight lines have been already defined. They, when we consider them as single lines, have length only; but a straight line may be of any length, known or unknown; and no straight line can be of a known length, unless we can measure it, and have measured it, or can deduce its length by correct reasoning from that of some line which we have measured.

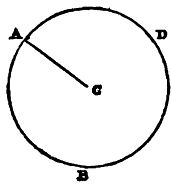
A straight line is represented geometrically by a line drawn as straight as possible; and it is named by two letters, placed either one at each end of the line, or at any point in it; and the line is named by those two letters, it being of no consequence which of them is mentioned first.

In geometry it is customary and advisable to use capital letters, as a distinction from the letters used in algebra, just as in algebra we use italic letters to distinguish them from the Roman letters generally used in printing common language. But when a geometrical magnitude is affected by a number stated generally by means of a letter, it is customary to use small or lowercase letters, and generally to use Roman ones, to distinguish them from the algebraical representations of quantities, just in the same manner as it is desirable to use Roman letters for exponents in algebra. We give an instance of the representation and naming of a straight line geometrically; thus, the following line is the line A B, if viewed from left to right, or the line B A, when viewed from right to left, but it is exactly the same line both ways:—

A——————————B

A *circle* is a plane figure, or portion of surface, bounded by one line, which is called the circumference, and the property of the circle by which it is defined, and from which all its other properties are derived, is that the circumference is everywhere equally distant from a point within the figure, which is called the centre of the circle.

The circumference, which means the measure round, or, literally, the "carrying round," and sometimes the periphery, which has the same meaning, is often called a circle, as well as the surface which it incloses. The circumference is named by any number of letters more than two, marked either without it or within; the centre by a letter marked as near to it as possible; and the surface of the circle either by the letters which mark the circumference, or by any letter within the figure. In the meantime we are considering the circumference only, and the relation which it has to the centre, namely, that above stated, that of being everywhere equally distant from it. Thus the following is any circle, A B D, of which C is the centre:—



The property of a circle, upon which its definition is founded, follows immediately from the way in which the circle is drawn or described. Thus, suppose a bit of thread, a bit of stick, or anything else of a constant length, as the line A C, has one end made fast at the centre C, and being kept perfectly and equally stretched so as to represent a straight line, and has its other extremity A carried round, either by B and D, or by D and B, till it comes back to the position A, the circle will be described; and if a pen, a pencil, or anything else that will leave a mark, is carried round at the point or extremity A, and made to mark a plane surface, a circle will be drawn upon that surface.

The line C A, extending from the centre to the circumference, is called the *radius*, or *ray* of the circle; and it is evident, from the manner in which the circle is described, that the magnitude or size of the circle depends upon the length of the radius.

It follows from this, that, as the radius is simply a straight line, which has no property but length, and the length of which is the same at every part of the circle, the circumferences of different circles can vary only in the same proportion as the lengths of the radii; that is to say, if the radius is double, the circumference must be double; if the radius is three times, the circumference must be three times, and so on in all other proportions.

In practice, circles of small dimensions are usually drawn with an instrument called a pair of compasses, the two points of the compasses being set at exactly the same distance from each other as the radius of the intended circle; and this is an instance in which the representation of a line by its extreme points, answers the same purpose as the line itself; from which we may conclude generally, that, if the two points which are the extremities of a line are determined, the line itself is determined.

In describing a circle with compasses, it is necessary that the distance between the points should remain exactly the same, otherwise the fundamental property of the circle is departed from, and there can, in fact, be no circle.

The postulate, or operation assumed as being self-evidently possible in the case of circles, is, that "a circle may be drawn from any point as a centre, and at any distance from that centre." The word "any," in both parts of this postulate, includes all points and all distances which we can by possibility imagine; and it is not confined to circles which we can actually draw or describe in practice, and show them after they are drawn. Hence we have a distinction between geometrical possibility and practical or mechanical possibility. Geometrically, it is possible to take the sun as a centre, and imagine a circle to be drawn passing through the most distant star which we can fancy—the distance of the sun and star, immeasurable as it is, being the radius of the imagined circle, and the circle itself will be more correct than any which we could actually describe by means of any instrument; but, in practice, no power of man could describe a circle of even a single mile in radius, simply because no power of man could keep an instrument a mile long on the stretch, and carry it round at the same time.

Geometrical possibility, or (for we may state it generally) mathematical possibility, extends to everything which is possible in thought, however impossible it may be, practically or mechanically, to the hand or the instrument. This is a simple consideration, but it is both a necessary and an important one; for there are many things in geometry, and in other parts of mathematics, which we assume as being done, though in the cases before us we could not possibly do them; and when we know that mathematics reach every thing which can be the subject of thinking, and give the precision of mathematical science to our reasoning upon all subjects, we are better able to appreciate the value of this science, and to profit by the appreciation.

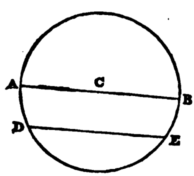
2. Of Surfaces or Figures.

We have already given a general definition of a plane surface, and therefore all that remains for us to do is briefly to define a few of the kinds or varieties of plane surfaces, and to point out the means by which they are represented geometrically.

In plane geometry, there are two general divisions of surfaces; first, the circle, of which some mention has already been made; and, secondly, rectilineal figures, of which there may be an endless variety, both as to the numbers of straight lines which form their boundaries, and as to the lengths of those lines as compared with each other. To these, however, may be added plane figures which are bounded partly by straight lines, and partly by circles.

As the boundary of a circle, which is one uniform curve, and is formed by one end of the radius being carried round, while the other end remains at the centre of the circle, is called the circumference or the carrying round the whole boundary; and the sum of the lengths of all the sides of any rectilineal figure, is called the perimeter, or measure about the figure.

The circle has already been mentioned, and the way in which the surface of it is named; but there are one or two particulars which are worthy of notice. Thus, if a line is drawn through the centre to meet the circumference both ways, it is called the *diameter*, or measure through the circle; and as this diameter is evidently double the radius, and as the radius is the same in all parts of the same circle, it follows that the diameter is always of the same length in the same circle, in what direction soever. Thus, in the following circle A D B E, of which C is the centre, the line A B, drawn through the centre C, and meeting the circumference in A the one way, and in B the other, is the diameter, and a line in any other direction, drawn through C, and meeting the circumference at each extremity, would, in the same circle, be equal to the line A B.



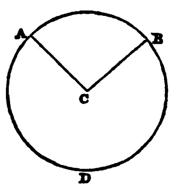
The diameter, A B, or any diameter which could be drawn in any other direction, evidently divides the circle, both circumference and surface, into two equal parts, each of which is called a *semicircle*, which is only another name for a half circle. That a diameter divides both the circumference and the surface equally, is evident without any proof, because there is nothing affects the portion of either, on the one side of the diameter, which does not equally affect the portion on the other.

If a line, not passing through the centre, is drawn till it meet the circumference both ways, as, for instance, the line D E in the above circle, it is called a *chord*, and the portion of the circumference which is cut or marked off by a chord, is called an *arch*, or *arc*. Thus, the chord D E cuts the circumference of the above circle into two arcs, a greater one surrounding the portion of surface in which the centre is situated, and a smaller one, in which the centre is not situated.

The portions into which the surface of a circle is divided by a chord, are called *segments*; and when a circle is divided into two segments, one is always greater than a semicircle, and the other less.

It may not be improper to mention here that the word *area* is often used for *surface*, and that the two words have exactly the same meaning.

A circle may be unequally cut by two radii, as well as by a chord, and in this case the parts into which it is cut are called



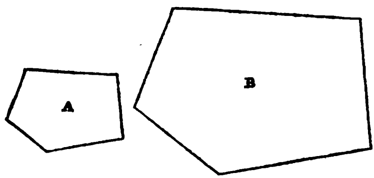
sectors. Thus, in the circle A B D, either of the portions divided off by the radii, A C and B C, opposite to D, and towards D, is a sector.

The circular part of the boundary of a sector is called an arc, as well as that of a segment; but there is this difference between them, that a segment has only one straight boundary, while a sector has two; that the straight boundary of a segment may be any line less than the diameter, but that the two straight boundaries of a sector, being each equal to the radius, are always, both together, equal to the diameter: and that, while a sector always extends to the centre of the circle of which it is a sector, a segment never does. A segment, too, is never more than a two-sided figure, the one side a chord, and the other an arc; a sector is always a three-sided figure, one of the sides being an arc, and two being radii.

Rectilineal figures are named from the number of their sides or angles, the number of sides and of angles in every rectilineal figure being equal. It is easy to understand why this must be the case: every side has two extremities, and every angle is formed by the meeting of one extremity of each of two sides. Of course no rectilineal figure has fewer than three sides and three angles, because three is the smallest number of straight lines that can inclose a space or surface.

Figures with three sides are called *triangles;* those with four sides, *quadrilateral* figures, or *quadrilaterals;* and those with more than four sides, *multilateral*, or many-sided; the last are also sometimes called *polygons*, or many-angled figures; but that name is, perhaps, better restricted to one particular form of figure, whatever may be the number of sides.

There are thus three particulars in all rectilineal figures having the same number of sides and angles, in which one figure may agree with or differ from another. There are, first, the magnitude of the angles taken in the same order; and when the figures have all their angles equally taken in this way, and all their sides taken in the same order, in the same proportion, the first to the first, the second to the second, the third to the third, and so on, the figures are said to be *similar*, which means that they are all of the same shape. Thus, the following figures, A and B, are similar; for all their angles, taken in the same order, are equal, and the sides of the figure A are, to those of the figure B, in the proportion of 1 to 2; that is, the sides of B, taken in the same order as the sides of A, are each twice as long.



The other relations which similar figures bear to each other can be better explained afterwards.

Rectilineal figures, which have all their sides and all their angles equal, are called *polygons*, or *regular* polygons; and if the name polygon is applied to a figure which has not all its sides and angles equal, the name polygon is qualified by the epithet irregular.

A three-sided regular polygon is called a *trigon*, or, more generally, an *equilateral triangle;* one with four sides is called a *tetragon*, or, more frequently, a *square;* one with five sides is a *pentagon;* one with six sides, an *hexagon;* one with seven sides, a *heptagon;* one with eight sides, an *octagon;* and so on, the name being compounded of the Greek term for the number of sides or angles, and the Greek name for angles.

Plane triangles are also distinguished into three kinds:

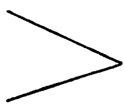
equilateral, having all the sides equal; *isosceles*, having two equal sides; and *scalene*, having all the sides unequal. *Isosceles* means having "equal legs," the third, or unequal side being called the *base;* and *scalene* means having unequal sides.

Four-sided figures are also distinguished into several species; but the nature of them can be better understood afterwards.

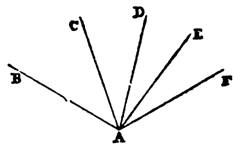
3. Of Angles.

The general nature of a plane rectilineal angle has been explained in the preceding section, and the comparison of angles with each other, together with their measurement, and the standard by which they are measured, will be explained afterwards; so that all that requires to be done in this place is to mention how an angle is represented and named geometrically. Now an angle is represented by two lines which meet at a point, the point where they meet being called the *angular point*, or the *apex*, or the *vertex* of the angle.

If there is only one angle at a point, it may be named by a single letter at the point, as, in the following figure, we would say, "the angle A," or "the angle at A."



But if there are several lines which meet at a point, then there are more angles than one, and it becomes necessary to place a letter on each line, at some distance from the point; and when we name any of the angles, we name, first, the letter on one of the lines, next the letter at the point, and thirdly the letter on the other line.



In the above figure there are five lines which meet at the point A, and thus there are four distinct and separate angles, all angles at A. Not only this, but there are as many more as can be formed of combining those in juxta-position, or taking those which are beside each other.

Taking the single ones from left to right, they are the angles BAC, CAD, DAE, and EAF, four angles.

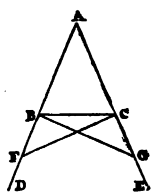
Next, taking them two and two, there are BAD = BAC + CAD, CAE = CAD + DAE, and DAF = DAE + EAF, three angles.

Again, taking them three and three, there are, BAE = BAC + CAD + DAE, and CAF = CAD + DAE + EAF, two angles.

Lastly, there is the whole angle, BAF = BAC + CAD + DAE + EAF, one angle.

So that these five lines meeting at the point A, form ten distinct angles. Cases in which there are more than one angle at the same point, require some attention from beginners, in order that they may avoid confounding the one with the other; and no small part of the difficulty which is felt in the case of complicated diagrams, arises from not having sufficiently studied the simple parts of which they are made up. "Take time, and get on fast," is no bad maxim in most matters, and there is none in which it is more applicable than in mathematics. Folks do not understand books without first knowing letters and words, and yet this method is not unfrequently attempted in Geometry, in Algebra, and in Arithmetic.

It often happens, in complicated diagrams, that the same angle belongs to more than one figure, and this is also a source of annoyance to beginners. As an instance of this, we may mention the fifth proposition of the first book of Euclid's Elements, the far-famed *pons asinorum*, or asses' bridge, the demonstration of which is very simple, as well as beautiful; but there is a perplexity in the angles, one of which belongs to three triangles; and of two other sets of angles, at two points, one belongs to one triangle, a second part to a second triangle, the third and second to a third triangle, and the second with the third on to a fourth triangle. As the diagram is a good study for those who wish to understand such representations, we subjoin it, and append the several triangles, which the reader can easily trace. The following is the diagram as it appears in the book:—



The object here is to prove that, if the sides AB and AC, in the triangle ABC, which comprises the upper part of the diagram, are equal, the angles at B and C must also be equal; and all that is admitted to be known about triangles is, that, if two sides, and the angle included between them in one triangle, are equal to two sides and the included angle in another, the two triangles are equal in every respect; that is, the third side of the one is equal to the third side of the other, and the remaining angles which are opposite the equal sides are also equal.

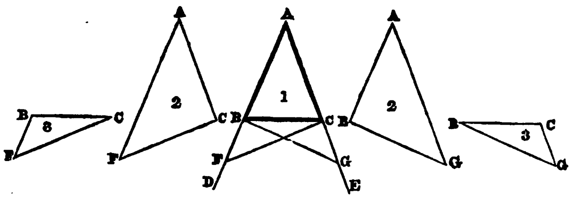
It must be admitted that the truth which is to be proved in this case, comes as nearly as possible to a legitimate deduction, or *corollary*, as it is called, from the truth by means of which we are to prove it; for if, in *two equal triangles*, the angles opposite to the equal sides are necessarily and in every case equal, it seems to follow that, as *one triangle* is in every respect equal to itself, the angles opposite to equal sides in it must be equal.

But though this would be a sound argument in ordinary reasoning, it does not come up to the rigour of geometrical demonstration, and so we must have equal triangles to compare with each other. For this purpose AB is extended to D, and AC to E; the parts BF and CG are taken equal to each other, and BG and CF are joined, which give four additional triangles, which, taken two and two, are equal to each other.

But this, though true, is not apparent to one unacquainted with diagrams. Only three additional triangles are apparent in the diagram, and we are not in possession of the means of proving that any two of them are equal in any respect; and though we were, they could prove nothing respecting the triangle ABC, for they are all external of it, and quite unconnected with the angles ABC and ACB, the equality of which is to be proved.

But let us analyse the diagram, and see what other triangles we can get out of it, without altering the relative positions of any of the lines.

The following figures contain the real and palpable analysis, which the student is called upon to make virtually, at the same time that he is making his first infantine attempt to wrestle with the giant of geometry:—



The middle figure will be perceived to be exactly the same as the diagram given above, the given triangle of which the angles at B and C are to be proved equal to each other. We have marked this triangle with the number 1, and strengthened the sides of it in order to distinguish it from those other triangles which are made for the purpose of the demonstration,—and as it should be in cases of teaching the first elements of geometry, where it is done by diagrams ready made, and not constructed (as they always should be when it it is possible) in the presence of the student. The two triangles marked 2, to the right and left, can be traced as answering to two equal ones which lie across each other in the central diagram, and have at their angles respectively the letters ACF and ABG. So also the two triangles farther to the left and right, marked 3, can be traced as corresponding with the two triangles in the diagram which lie partly across each other below the side BC, or base of the original triangle. The triangles 2 have each the angle at A equal to the angle at A in the original triangle, and their sides, CA and AF in that to the left, are equal to BA and AG in that to the right, and therefore they are equal in every respect, and FC and BG are equal to one another, and so are the angles at C and B, and also those at F and G.

But the trianeles 3, 3, have BF and CG made equal, and FC and GB proved equal, and also the contained angle at F equal to the contained angle at G; therefore they are equal in every respect, and the angles opposite equal sides are of course equal, that is, the angle C in the left-hand one is equal to the angle B in the right, and the angle B in the left-hand one is equal to the angle C in the right. But the angle C, of triangle 2 on the left, is equal to the whole angle ACF in the diagram; and the angle F, in triangle 3 on the left, is equal to the part BCF in the same. So also the angle B, in triangle 2 on the right, is equal the whole angle ABG in the diagram; and the angle B, in triangle 3 on the right, is equal the part CBG in the diagram. Now,

From	ABC =	ABC + CBG, and	= ACF =	ACB + BCG
Subtract		CBG and		BCG
There remains		ABC	=	ACB;

and they are the angles opposite the equal sides, or at the base of the given triangle.

Again, FBC, in triangle 3 on the left, has been shown equal to GBC in triangle 3 on the right, and they are respectively equal to FBC and GCB in the diagram; and these last are the angles on the other side of the base, formed by the base and the equal sides produced. Therefore, if a triangle has two equal sides, the two angles at the base, and the two angles on the opposite side the base, are equal to each other.

In geometrical language, the words "each to each" are made use of for shortness of expression, when any number of pairs of quantities have each pair equal to each other.

In the analysis of this diagram we have rather anticipated, in introducing the demonstration, but a very little attention will suffice for understanding it; and the analysis of the mere diagram would not be so well appreciated, unless the use of it were shown at the same time.

The ready understanding of diagrams, so as virtually and at a glance to dissect or analyse them into all the parts of which they are made up, is a most essential qualification in those who wish to understand easily even the simplest elements of geometry; but it is almost, or altogether, omitted in the books, and the omission is, in our opinion, chargeable with much of the perplexity and failure which so many meet with in this science. It would be too much to suppose that we have furnished, in this section, the means of wholly overcoming the difficulty; but if what we have stated is read with attention, and some practice is taken with the analysis of diagrams, either in Euclid's Elements, or in any other elementary work, the the student will be enabled to proceed to the work of investigation with much more ease and prospect of success, than if he were not so prepared. We shall examine the principles and processes of geometrical investigation in the next section.

SECTION XI.

PRINCIPLES OF GEOMETRICAL INVESTIGATION.

INVESTIGATION, taken in a general sense, means systematic and accurate inquiry, in order to determine whether something which is alleged is true or not true, or whether something proposed to be done is possible or not possible; and of course, geometrical investigation includes every instance of both of these, of which the subject can be considered as geometrical, that is, as being a magnitude, or capable of being exactly or nearly estimated in some known measure of the same kind.

A subject which is proposed for geometrical investigation is in general styled a *proposition*. If the object is the establishment of a truth, the proposition is called a *theorem;* and if it is the performance of some operation, or the obtaining or determining of some unknown quantity by means of known ones, then the proposition is a *problem*. Either of these, however, equally admits of proof; for when the problem is solved, or the unknown quantity arrived at, it is necessary to show, not only that it is the quantity which is required, but that it has been fairly arrived at, by legitimate reasoning, founded on the conditions which are given; and both in this case and in proving the truth of a theorem, if the data or conditions given are not sufficient, or if there are not sufficient means of connecting them with the conclusion, then the problem will remain unsolved, or the truth of the theorem will remain unestablished.

When a general truth is once established, one or more subordinate truths often arise from it by inference, and without the necessity of that laborious investigation which the original truth requires; and these subordinate truths are called *corollaries* to the general proposition on which they depend.

Different branches of geometrical science are usually taken in regular trains, so that the full investigation of one subject may be clear and connected, and without intermixture with other subjects; but as all the branches of geometry, and indeed all the branches of mathematics, whether geometrical or not, are connected with each other, and one often assists greatly in the investigation of the other, it is sometimes necessary to break in upon the regular succession of propositions on one subject, or to preface them by a proposition which in part belongs to another subject. A premised or interpolated proposition of this kind is called a *lemma;* and as the object of it is the introduction of a collateral truth, which shall assist in the general investigation into which it is brought, it usually partakes of the nature of a theorem, and as such stands in need of demonstration. As the introduction of a lemma breaks the regular succession of the principal matter, it should not be introduced unless when the advantage to be derived from it fully compensates the departure from strict regularity; but this is more worthy of attention in systematic mathematics, where elegance is attended to as well as usefulness, than in the merely popular knowledge of the science.

In addition to these different forms in which mathematical truth (for they apply to numbers, and to quantity generally, as well as to magnitude) may be introduced, there are often explanations necessary, which do not amount either to propositions as distinct parts of the subject, or to lemmata as connecting points introduced from other subjects; and these explanations or illustrations are called *scholiums.*

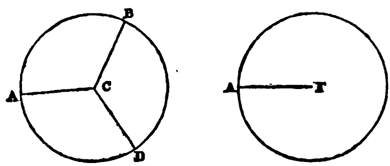
Such are the technical names of the chief divisions of geometrical investigation; and in the preceding section we have given some account of the principal kinds of geometrical quantities, namely, lines, angles, surfaces, and solids; and the definitions of these, both generally and in their several species, should always be founded upon their most obvious property, that which can be expressed in the fewest words possible, and which, while it is clearly and also sufficiently descriptive of that which it purports to define, should at the same time exclude every property which does not belong to the thing defined. Thus, when we say "a line is length only," we express every property which belongs to a line in the abstract, and which, while it is descriptive of all lines, whether straight or crooked, excludes everything which is not a line. So also, when we say "an angle is the inclination of two lines to each other," without adding, "in the same plane," or "which meet at a point," we give a definition of angularity, or inclination, as contrasted with parallelism, which is perfectly general, because it includes every possible case of lines, of whatever form or extent they may be, which are nearer to each other at some parts of their length than at other parts.

The most careful attention to definitions is indispensable to every one who wishes to profit by the study of geometry; but there is nothing more dangerous to a beginner in the science than committing to memory the mere words of definitions, however accurate, or however well expressed. No one ever really arrived at the knowledge of a subject by this means; and it is impossible to say how many, resting satisfied with the mere "parroting" of the words, have never made the slightest effort to comprehend their meaning. The best plan is to take the subject apart from other subjects, and form one's own knowledge of it; and whatever may be the words in which one may be able afterwards to express the definition, the reality of it is sure to be impressed on the mind; which is by far the most important part of the matter.

Besides the definitions of quantities, there are certain general grounds of belief, and certain assumed performances of operations, which are necessary before we can come to the investigation of even the simplest geometrical truth, or succeed in the performance of even the simplest geometrical problem. Those fundamental grounds of belief are called *axioms*, which means that the truth of them is self-evident, such as must be admitted by every one who understands the words in which they are expressed. The number of such axioms introduced by different writers on elementary geometry differs, but in substance they are nearly the same.

The operations, the practicability of doing which is understood to be obvious to everybody, are called *postulates*, and they are usually restricted to the three to which allusion was made in the last section—the drawing of a straight line from any point to any other point, the producing of a terminated straight line to any greater length still in the same straight line with the former part, and the describing of a circle from any centre with any radius. In these postulates it is not to be understood that the operation is to be mechanically or actually performed, because that could not be done in any one of the three cases; but all that is meant is, that there is no geometrical absurdity in supposing any of these things done; and that though the line were joining the sun and the most distant planet, the extension this line produced to the most distant star, and the circle described with that produced line as a radius about the point where we stand as a centre, we have just as clear a conception of those mighty lines, and that mighty circle, as we could possibly have of a line drawn on paper, from one dot to another an inch apart, of the extension of this line another half-inch, or of the description of a circle round any dot on the same paper, with a radius or extent of one inch between the points of an ordinary pair of compasses.

In the postulate that "a circle may be described from any centre, and at any distance from that centre," there is involved a far more general postulate, and one which, when stated with some explanation, tends greatly to simplify the investigations of elementary geometry. In the first place, the describing of the circle assumes, or takes for granted, that the line with which the circle is described, that is the radius, can be placed in every possible direction, if it has one end always at the point; for this is the very property upon which the definition of the circle is founded. Thus, in the circle ABD, of which C is the centre, and AC the radius, that radius may also have the position CB or CD, or any other position, so that the one extremity of it is at the centre, and the other at the circumference. Not only this, for, in the second place, a circle may be described with the same radius, CA, from any other point as a centre; as, for instance, from the point in the second of the annexed figures; and if AF, in the one of those circles, be exactly equal to AC in the other, it is self-evident that the two circles must be every way the same; and that this will hold true not only of them, but of all circles having equal radii, wherever the centres of those circles may happen to be situated; and if the circles are in every respect equal, it must follow conversely that all radii that can be drawn in each or in all of them, in what direction soever they may be situated, must also be equal to each other.

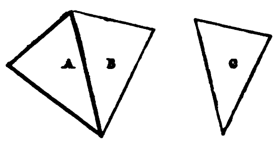


It is easy to perceive that this may be generalised so as to include all lines, and consequently all figures, and all solids; for it is self-evident that, if any line whatever can be supposed to be placed in any situation, and in any direction, everywhere throughout absolute space, and be of exactly the same length in every possible situation and position; then, whatsoever can be applied to that line in any one situation or position can be applied to it in every other situation and position; and if applied to exactly the same extent, and in exactly the same manner, the result of the application must be the same in every possible case, and also the same whether applied to the one side of the line or to the other.

We may therefore assume as a postulate, that "if any figure is already applied, or can be supposed to be applied to any straight line in any manner, a figure exactly equal in every respect may be applied in the same manner to any other straight line equal to the first, whithersoever that line may be situated."

And from the same principle we may conclude with equal certainty, that, "if a figure is applied to one side of a straight line, a figure equal in every respect may be applied to the opposite side of the same."

If figures of this last description are not regular, that is, if all their sides, and all the sides and angles of each, are not equal, they are *symmetrical* magnitudes, that is, magnitudes of "equal measures," but reversed in their position with regard to each other; thus, if a diameter is drawn across a circle, the two semicircles, into which it divides the circle, are exactly equal, but they are symmetrical, as their circular sides are turned in opposite directions; also, if one triangle is constructed upon the one side of a line, and a triangle equal in all respects is constructed on the opposite side of the same, or of another line, such triangles are symmetrical: as, for instance, the triangles B and C are symmetrical with the triangle A, B being constructed on the opposite



side of the same line with A, and C on the opposite side of a different line; but all these three triangles are in every respect equal to each other.

The body of a perfectly formed animal, if we imagine it to be divided exactly on a mesial plane, that is, a plane extending the whole length of the animal, and passing exactly through the middle of the upper and under parts, affords a very good instance of what is meant by symmetrical magnitudes, as applied to solids. The two portions into which the animal is thus supposed to be divided, are exactly equal to each other in every respect; and yet we cannot, even if we were actually to divide the animal, place them so as to have them in the same position at one view, and thus judge in detail of the perfect equality of all their individual parts; for if we placed one of them in the natural position of the entire animal, we could not show the external surface of the other one without turning it either end for end, or upside down in respect of the first. Many cases of symmetrical magnitudes occur in investigations which are purely geometrical; and therefore, if we are not aware of them beforehand, we are apt to feel less certain in our reasonings respecting them, than we are respecting magnitudes which present themselves to us in the same position with each other.

The admission of the postulate which we have mentioned, or rather of that necessary inference from the third postulate usually given in the elements, completely obviates this difficulty; and, when carefully considered, there is an axiom deducible from it, or rather arising necessarily and obviously out of it, which enables us to get the better of many difficulties. The axiom to which we allude, when stated in its most general terms, is as follows:—

If we know with certainty all the circumstances upon which any two conclusions or results depend; and if we farther know that those in the one case are exactly the same as those in the other, each to each, in the same order; then the two results or conclusions, whether they be truths which are established, or figures which are constructed, or quantities which are found, or, in fact, any results whatsoever, must be exactly the same.

It is true that this axiom has not the apparent simplicity of the usual axioms in elementary geometry, but the fact is that it embodies them all, and a good deal more, some of which has to be demonstrated, and some is taken for granted in the course of the elements.

All mathematical reasoning is by comparison of quantities of the same kind; and the conclusion arrived at in every single step of such comparisons, is the equality or the inequality of the quantities compared; for if we seek to find the difference or the ratio, the determination of this requires a second step, and that step is arithmetical in any one particular case—an instance of subtraction, if we seek the difference, and an instance of division, if we seek the ratio. Therefore, it becomes necessary that our original notion of equality should be such as to embrace all possible cases.

Now, the fundamental axiom usually given in elementary geometry, though the eighth in order, and not the first, is in these words;—"Magnitudes which coincide with one another, that is, which fill exactly the same space, are equal to one another."

This axiom, by the introduction of the word "magnitudes," not only limits the case to geometrical equality, but it actually does not reach nearly to the whole of that. In strict language, no magnitude but a solid can be said to "fill space," for a line occupies no space, neither does a surface, unless when we regard it with relation to a solid. A line may, no doubt, be called a magnitude, but then it is a magnitude of one dimension only; and a surface is but of two dimensions, while it is just as impossible to imagine the existence of space without three dimensions, as it is to take up in one's hand a piece of board which has no thickness.

Farther, an angle is not a magnitude which fills space, and indeed it is not, strictly speaking, a magnitude of any kind; because, without the imagined existence of two lines which have an inclination toward each other, there cannot be even an imagined angle. In like manner, no ratio can be regarded as a magnitude capable of filling space; and therefore it is that the doctrine of the equality of ratios, as expressed, and clearly and beautifully expressed, in the fifth definition of V., Euc. El., is so difficult to every student, and proves an insuperable barrier to so many.

Now, the doctrines of lines, of angles, of surfaces, and of ratios, are quite as essential in geometry, as the doctrines of those magnitudes which can fill space, and of which the equality can be established by its being shown that they fill the same space. They are even more essential, because they are the elements by the relations of which to each other the form and extent of any magnitude which can fill space are determined; and therefore, either the nature of those lines, angles, and surfaces which are compared in the earlier parts of the elements, are not understood, or the truth of the axiom we are considering is tacitly assumed, without being stated, which is certainly a very ungeometrical method of proceeding.

But there is another advantage in taking the doctrine of equality at once in its most general form, which is of much more importance to us than anything in mere geometry, important as that branch of science is. This very axiom is our general, we may say, universal rule in all our reasonings, and all our actions; in every department of science, or subject of knowledge, be it what it may, and in every action of our lives, if we act as rational beings, that is, if we have an object in view, and seek to accomplish that object by the most simple and most certain means. We proceed upon what is usually called the judgment of experience; that is, we observe personally, or we are informed upon testimony which we have no reason to doubt, that formerly certain data, or things, or circumstances known to us, applied or acting in a particular manner, to a particular extent, in the former cases, produced or led to a certain definite result; and upon the faith of the axiom (or maxim, as we call it in matters of real life), that, "in like circumstances a like result must take place," we pursue our artificial plan with confidence of success, and therefore with pleasure; or, if the result be one which must be brought about by natural causes, with or without our assistance, we wait that result with the patience of wisdom, and do not spoil by attempted hurrying, that which, in the nature of things, we cannot hasten.

If, in our acting upon this maxim, we could obtain a perfect knowledge of all the circumstances, that is, of all the data, and all the means of dealing with this data, our expected results would all be physical or moral certainties; and though, even in physical matters, we cannot exactly accomplish this, we can always do it the more nearly, the more completely that all the circumstances are known to us. Thus, for instance, because the data are few, well understood, and, generally speaking, reducible to mathematical laws, we can notwithstanding the many variations in the motion of the moon, tell what shall be the apparent distance of that luminary from any fixed star, at any time, long before that time arrives; or we can, in the case of an eclipse of either of the great luminaries, predict the moment of its commencement and termination, and the portion of the luminary which shall be eclipsed, for almost any number of years before the eclipse takes place. In the case of a comet, our calculations are not quite so accurate, or so much to be depended on, because there is one element in the case of those more flimsy bodies with which we are not so well acquainted. This element is the quantity of matter in the comet, by means of which alone we can determine the reciprocal disturbance between it and any other body whose motions are more regular, and whose mass is known to us. But even in this case, the application of mathematical principles, and of this axiom among the rest, has enabled us not only to remove that superstitious dread of comets which so much alarmed the ancients, but also to get rid of that alarm at the possible collision of our earth with one of those wanderers, which was a source of some apprehension to speculative men during the middle ages, or in the more early days of modern science. In matters of geometry especially, but generally in all branches of pure mathematics, that is, where natural causes and human actions do not enter into the case, we have the whole data, and also the whole management of that data, completely under our controul; it is "our own" in a far more personal and intimate manner than any possession, or any enjoyment of the body; and therefore we may "do with it as we list," provided we do not violate those laws which are the very foundation of this description of knowledge.

But, again, there is yet farther this advantage in the general doctrine of equality over the partial one of "magnitudes coinciding, or filling the same space," that it applies to and includes equality of process or operation, as well as equality of ratio and equality of magnitude; and this is a very important matter, because, from the definitions we have already given of lines, surfaces, and solids, and also from the few hints which we have thrown out respecting ratios (and which we shall resume and treat more at length in a section expressly on the subject), we may see that the result of a mathematical operation is often a quantity of a kind totally different from any of the given quantities by means of which the operation is performed, just as a chemical compound may be, in all its appearances, and in all its useful properties, totally different from one and from all of the ingredients of which it is the componnd, so long as these remain unmixed with each other, or unmixed by the very process which we employ.

Thus, if we seek to know the contents of a solid, as, for example, the number of feet in a log of timber, we do not go about to apply a solid foot to it, and see how many repeated applications of this solid foot coincides, or fills the same space with the log. This, in fact, would be impossible by any direct comparison, because, although we had a standard which we knew to be exactly equal to a cubit foot, we could not get this cubic foot and a cubic foot of the log into the same cubic foot of space, without previously removing the foot of the log; and though by this means we might show that the one was equal to the other, that is, that the foot put in occupied exactly the same space with the foot taken out, yet the information thence arising would simply be, that a cubic foot is equal to a cubic foot, which is really nothing.

In order to compare the log with the cubic foot, that is, to tell the number of feet in it, we must cease to consider it as a real and tangible solid, and regard it as a mere relation of three lines—the length in feet, the breadth in feet, and the thickness in feet; and we must find the lengths of those lines, not as having any connection with the solid, but as being the shortest distances between their own extreme points. Thus, if the length of the log is 12 feet, the breadth 3, and the thickness 2, we have, from the relation in which these lines stand to each other in the solid, the log = 12 × 3 × 2 solid feet = 72 solid feet.

From this we derive not an unimportant distinction of when quantities, expressed in numbers, or by the more general symbols of algebra, are or are not, geometrically speaking—indeed, mathematically speaking—quantities of the same kind, and thus whether they are or are not fit subjects of comparison, or such as can have a ratio to each other, either of equality, or of inequality. We have already alluded to this subject, but there are some mathematical considerations which belong so equally to different branches of the science, and which serve to connect those branches with each other, and with the practical business of life, so usefully, that it often becomes necessary to bring a truth which has been already examined into juxtaposition with a new truth, in order to point out the connection or relation between them; and, as much of the clear and ready understanding of the whole mathematical sciences, especially in their connections and their applications, depends on the clear perception which we have of this doctrine of equality, of the means by which it may be shown, and of the changes that may be effected on quantities without destroying it, we have been anxious to treat this subject very fully, even with the certainty that it must appear tedious to those who are already acquainted with it.

Mathematically, simple numbers, that is, numbers which are not considered as the results of any multiplication, are always regarded as represented, that is, as being capable of representing lines only. Products of two factors are considered as representing surfaces, and products of three factors are considered as representing solids. Magnitudes cannot be more than solid, and therefore there can be no more geometrical magnitudes answering to the products of numbers than these three; but still, generally speaking, products which arise from multiplying equal numbers of factors, that is, which are produced by equal numbers of multiplications, are always considered as quantities of the same *degree*, if not absolutely of the same kind, and therefore they are comparable with each other.

After we once fully understand the general doctrine of equality as applicable to all quantities of the same kind, and to all changes or operations which are equally performed on them, it is of some advantage to enumerate the particular cases, if only for the purpose of ready quotation, in those instances to which these cases are applicable; and in this respect the coincidence of magnitudes which can be superposed, or applied the one upon the other, and shown to be co-extended when this is done, may be admitted as quite satisfactory. Lines and plane figures are almost the only ones which can be compared in this way; and straight lines are equal when it can be shown that the extremities of the one coincide with the extremities of the other; and as they have no respect in which they can be equal as magnitudes, save length only, if the points which make the extremities of one can be shown at the same distance from each other, as those which mark the extremities of another, the equality will follow as a matter of course, without applying the one to the other.

There is also an indirect method of showing the equality of lines, and that is, by proving by reasoning that the one cannot be either greater or less than the other. This applies to quantities generally, and the method of proof generally turns upon some absurdity which would be the result of any quality in the two quantities, which are by this means proved to be equal.

In the case of surfaces, if, upon the application of the one surface to the other, it can be shown that all the lines which are the boundaries of the one coincide exactly with all those which are boundaries of the other, then it will follow that these surfaces, and also their respective boundaries, both sides and angles, are every way equal.

Also, two quantities of the same kind are equal, if it can be shown that each of them is equal to some third quantity, or to any equi-multiples, or like parts of a third quantity—by "equi-multiples" being understood all possible products by the same multiplier, whether that multiplier can or cannot be exactly expressed by arithmetical notation; and by "like parts" are understood all possible quotients that would arise from dividing by the same divisor, whether those quotients can be accurately expressed by single quantities or not.

This is the test, or judgment of equality, to which we are in the habit of appealing in practice, and according to which all equitable exchange of commodities, all estimates of the quantity of materials required for any specified purpose, and, generally speaking, all measurement or knowledge of the values of quantities, are determined; and therefore it is the case of the general question of equality, which everybody ought to understand best. Thus, for instance, if we have in England a certain standard of length, which we call a yard measure, and carry this measure to any number of different parts of the world, and by applying it to certain lines, or lengths, or breadths, which are fixed and immoveable in those distant places, and which, therefore, instead of being capable of being placed in juxtaposition, and judged of as filling the same space, or different spaces, cannot be both seen till after months, or perhaps years, have elapsed, and, generally speaking, which cannot be seen at all by the same individual, as certain that each of these is the same multiple, or the same part of a yard measure, we have no more doubt of their equality than if we could actually apply the one line to the other, and see their coincidence at both extremities.

We have the same confidence in such measures, even when they are not equal. Thus, if it is ascertained that one mountain, situated at the Himalaya ridge on the north of India, is 8,000 yards above the level of the sea, and that the height of another mountain, situated in the ridge of the Andes, in South America, which is nearly half the measure round the earth distant from the former, is also 8,000 yards, we have no more doubt of the equality of the height of those mountains, than if we could see them both side by side, and with our own eyes, at the same instant.

Without our belief in the fact, "that things equal to the same thing are equal to each other," we could not, geometrically speaking, have any map, any plan, or any pictured representation, whereby an absent thing could speak at once to the eye in that language which is so much more powerful than writing; and without the same belief in all other matters, we could have no knowledge, except that which we derive from our own senses; and even the parts of this knowledge would be unconnected, and resemble that which may be presumed to be the momentary perception of brutes, rather than the conclusions of human reason. Whenever we see an object of the same kind with one which we saw formerly, or one of which a clear description struck us forcibly, so as to make us remember it, whether we actually saw it or not, we certainly, and without any perceived or felt process of thought or effort of the mind, institute a comparison between the perceived object and the recollected one, and with as little effort we instantly conclude that they are or are not like or equal to each other. This is our primary and general judgment of equality or inequality, for the notion of the one involves in it a notion of the other. Mathematical or geometrical equality is merely the particular branch of this general judgment, in which we have the evidence most perfect in its kind, and most completely before us; and therefore it is only the most accurate case of that exercise of the mind which we must all practise every day.

Original and simple quantities may be equal by construction, that is, they may be made equal; and this is perhaps the best illustration which we have of equality, in the most general sense of the action, that is, where all the circumstances which determine the magnitude of the one, also determine the magnitude of the other, and every one of them is known to us as being our own act. Thus, if from any two centres, with the same straight line as radius, we describe two circles, it is impossible to have a more clear and simple notion of perfect equality than is afforded by those circles; and when we have this perfect perception of equality, it leads to perhaps the most general and the most important conclusion in the whole compass of mathematical science; and if this conclusion is not a direct axiom, it is as axiomatic, as self-evident an inference as can possibly be drawn. It is worthy of being borne in mind, and it is as follows:—

If two quantities, whether they be magnitudes, ratios, or anything else, are every way equal, whatever can be shown to be true of either of them, is necessarily true of the other, in the very same manner, and to the very same extent.

If the quantities of which the equality is asserted are results of any of the four arithmetical operations, we may state generally that equal operations, performed upon equal quantities, must produce equal results. Or, taking each operation—

In addition, if a is $= b$, and c any third quantity, whether a simple quantity, or one which is proved to be the same in two cases, then $A + C = B + C$.

In subtraction, the data being the same as for addition, $A - C = B - C$.

In multiplication, if $A = B$, and m any multiplier whatever, integral or fractional, expressible or not expressible in numbers, then $mA = mB$.

In division, if $A = B$, and d any divisor whatever, then $\frac{A}{d} = \frac{B}{d}$.

These, which follow both from what has been now stated, and what was formerly stated on the subject of multiples and fractions, may be quoted in brief thus:—The sums, differences, equi-multiples, and like parts of equal quantities, whether magnitudes or any quantities whatsoever, are equal. It is scarcely necessary to add, from what has been already said with reference to addition and subtraction, that equal operations must leave unequal quantities unequal; for if we add or subtract equally, we do not add or subtract any difference; if we multiply the unequals equally, we multiply the difference; and if we divide them equally, we divide the difference: but in none of these cases is the difference taken away.

As inequality is the opposite of equality, it follows that, in all cases where the one can be clearly established from a knowledge of all the circumstances, the other is proved in every case where it can be clearly shown that that one does not hold. But there are so many ways of expressing quantities, that we are not able, in all cases, to prove that equality exists, even where such is the fact; and therefore, our not being able to prove equality, is not, in every case, a sufficient ground for inferring that quantities of which we are unable to show the equality, are unequal. If we know *all* the conditions or circumstances upon which the values of two quantities depend, and if these circumstances, taken jointly, do not admit of the inference that the quantities are equal, we may safely infer that they are unequal; but if we are not sure that we are fully in possession of all the circumstances, we cannot conclude either way, in consequence of our failure in the other. Thus $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c., continued till the last term is $\frac{1}{\infty}$, that is, till the denominator is infinitely large, and the value of the fraction, $\frac{1}{\infty}=0$, is really $=2$, and merely an expression of a particular form for that number, as will be shown afterwards; but when we examine this, even to any extent which can be written down, it does not, upon mere inspection, appear to be $=2$. We mention this merely to show that an apparent inequality is not sufficient ground for inferring that the inequality is real, unless we can prove that we are in possession of all the circumstances upon which both the quantities under comparison depend.

In the case of quantities which are represented by products, the total value of the one may be equal to that of the other, though both the factors, in the case of there being only two, or all the factors, in the case of there being more than two, are unequal to each other; but if one factor in each be equal, in cases where there are only two, or if all the factors be equal except two, that is one in each, where there are more than two, the values of the products are unequal, and the difference between them is the difference of the unequal ones multiplied by the equal one, or the product of all the equal ones, in the case of there being more than one in each product.

Thus the product of the factors 4×4, 8×2, 16×1, is $=16$; and there are other cases in which, even in integer numbers, the same product may be obtained from a greater number of pairs of factors; and even in this case, the product 16 may also be considered as that of four factors, for $2 \times 2 \times 2 \times 2 = 16$, as before

If we examine the different factors which produce the same product in this case, we perceive a general principle, the nature of which will be more completely explained in a future section, when we return to the consideration of general quantities algebraically; but it may not be amiss to bear it in mind, without going into a full explanation of it. In the above numbers it will be seen that, when the factors are equal, their sum is less than when they are unequal, and that the more unequal they are, their sum is the greater. Thus, in the equal factors of 16, 4 and 4, the sum is 8; in the first unequal ones, 8 and 2, the sum is 10; and in the last, where one of the factors is the product itself, and the other the number 1, the sum is greater than the product, for $16 + 1 = 17$.

We mention this merely to show that, as two factors may be considered as representing length and breadth, and their product surface, the surface is always the greater, in proportion to the sum of the dimensions, that is, of the length and breadth, the more nearly that these are equal to each other. The same will also evidently hold true in the case of solids; but the consideration of these, as well as the investigation of the general principle of which the above product and its different factors are an instance, can be better explained in a future section, when we have a few more of the principles before us, so as to be able to consider generally the relations between factors and products, as compared with surfaces and their boundaries. What we have now said is sufficient to show that figures may be equal in content or area, though their dimensions, and consequently their boundaries, are altogether different; and it is necessary for us to bear this carefully in mind, in order to avoid supposing there is equality or inequality inferrible, in cases where no inference can be drawn. The inefrence of the area is always from the product of the length and breadth, and if these are known, the product or area is known; but if the product or area only is known, the factors of that product—the length and breadth—are quite indeterminate, only we may gather from the case of equal factors of 16, that the factors of any product cannot together be less than the two equal factors which can form that product, and that these two equal factors must be the same for the same product.

The doctrines of equality and its opposite, which we have endeavoured to explain in this section, in a manner the most general, and the most simple as well as comprehensive in its application, not only to mathematical subjects but to all subjects where a question of equality can occur, is one which requires to be studied with the greatest care, because it is the foundation of very much of our accurate judgment, upon almost every question that can be named as determinate in its evidence, and also our best security against error in cases which involve uncertainty. The reader who wishes to profit by this book will therefore find his advantage in giving this particular section a second perusal.

SECTION XII.

INTERSECTION OF LINES, ANGLES, AND SIDES AND ANGLES OF TRIANGLES.

After having obtained some general notion of the subjects of Geometry, as mentioned in section X., and the leading principles of Geometrical investigation, as in section XI., we have to consider the order of the subjects, and to take them in that which appears to be at once the most simple and the most natural. For popular purposes the following is perhaps as convenient as any.

First, LINES, including straight lines and also the circle, with the intersections of straight lines, the angles which these form, and the connection between plane rectilineal angles and the circle. This is the proper foundation of the science, and contains the elements of the boundaries of the more simple elementary figures.

Secondly, SURFACES, that is, *plane* surfaces, or areas, considered in their extent, and with reference to their boundaries. But as an area is determinable only by an arithmetical multiplication, in any particular case, and as, consequently, the general investigation is a matter of quantity and operation jointly, it will be necessary, before we proceed to this, to consider the doctrine of *proportion*, the *powers* of quantities, and the *arithmetic* of *exponents*, each of which will form the chief subject of a section; but, as they are all intimately connected, much reference from the one to the other will be required.

Thirdly, the INTERSECTIONS OF PLANES, by means of which the forms of plane solids are determined; and this will include the doctrine of SOLID ANGLES, or of more planes than two meeting in the same point.

Fourthly, the CONTENT OR CAPACITY OF SOLIDS, taken in conjunction with the planes which form their boundaries, and the lines and angles made by the intersections of those planes.

These will put us in possession of the principles of elementary geometry, as far as they are necessary in the ordinary business of life; and then we can return to the general science of quantity, and if our limits permit, to the applications. We shall devote the remainder of this section to the consideration of lines and angles, the simplest case of which is that of a point.

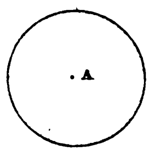
Now any point, as, for instance, the point at A,

.A

may be regarded as the centre of a plane, extending equally and immeasurably on all sides, so as to bisect or divide into two parts exactly equal to each other the whole of space. It is perfectly indifferent where we consider this point to be situated, because, as we can no more conceive or imagine a boundary to space in one direction than in another, we may suppose any point whatever, be it situated where it may, as being the centre of space, this point being a mark of position only, and having no extent in any direction, may be considered as equally the centre of the plane which bisects the whole of space, in whatever direction that plane is situated. According to our common notions, in which we regard a straight line directed to the centre of the earth as being the perpendicular, the plane may be in the direction of this perpendicular; it may be in the cross direction to this, or in what we call the horizontal position or the level; or it may be at any slope whatever, and may slope in any direction; but in all the endless variety of positions which we can with equal propriety suppose it to have, and in all the endless changes of position in the plane, we may still conceive the place of the point as remaining exactly the same in absolute space, and the plane extending indefinitely every way, but every way equally, and in all possible positions dividing the whole of space into two parts exactly equal to each other.

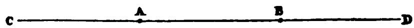
This notion of the perfect immovability of a point, and the possibility of turning a plane on this point in every imaginable position throughout absolute space, has not hitherto, we believe, been alluded to in books on elementary geometry; but it is, in truth, the grand primary conception, by means of which the geometer is enabled at once to lay his grasp upon the whole universe; and, seizing element after element as they arise, in due order and according to proper laws, to map down upon the tablet of his mind all that creation which God has made, as far as the line and the angle, magnified by the utmost perfection of the instruments of observation, can carry him.

As the space marked out by a plane round any point extends equally in all directions, the best representation which we can have for it is a circle, as, for instance, the circle of which the centre is the point A.

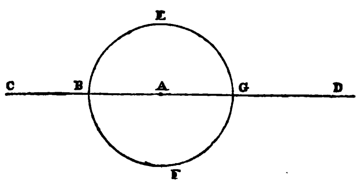


It will be recollected that the very definition of a circle is, that the circumference, or line bounding it, is in all directions equally distant from the centre; and therefore, if we imagine the radius, or distance from the circumference to the centre to be indefinitely long, the circle becomes the best representation which we can have for a plane extending through all space; and because the circle which we have described round the point A is *any* circle, we may regard its circumference as representing all space round the point A; and farther, as the point A is the only thing which is supposed to have *position*, that is, to be *fixed or determined* in space, we may regard this circle as the representative of all space in every possible direction, or that within it, in one or other of its possible positions, it can contain every line, every figure, and every solid which can by possibility exist in nature, or which the most fertile imagination can picture to itself. When, however, we refer to the circle in *one* position as a plane we can consider it as including plane surfaces only, and as including those only which are situated in the same plane with the circle.

Let us next suppose that there are *two* points, that they are both fixed in position, and that a line is drawn through them both, as, for instance, the line C D, which passes through the two points A and B, and is continued, or might be continued, to an indeterminate length, in the left hand direction toward C, and in the right hand direction toward D.



It is evident that, because the two points A and B are supposed to have *position*, that is, to have *fixed places in direction and in distance from each other*, the whole line C D as it appears, or as it could exist, though drawn countless millions of miles both toward C and toward D, is also fixed or definite in the direction of A and B, or of any other points that can be imagined to be taken in it. If now, then, we suppose a circle to be described round any point in the line C D, as, for instance, round the point A,



it follows that the plane of this circle must be confined to the line C D in the direction of whatever points it may cut this line, as, for instance, the points B and G in the above example; but it would be the same in the case of any other two points in the line

C D, or in the continuation of that line, how distant soever they might be from the point A. Hence it is evident that, supposing the line CD fixed to the plane of the paper on which it is drawn, we could not imagine the circle round A, to preserve that line, and be at the same time turned either to the right hand or to the left. But it is equally evident that we could turn it upon the line CD, either upwards in the direction of the top of the page, or downwards in the direction of the bottom; and that we could turn it round again and again in either of those directions as often as we chose, without in the least disturbing the position of the line CD.

It is further evident that the line CD bisects or divides into two parts exactly equal, the whole of that indefinite space through which it is supposed to be indefinitely drawn; and that if we take any circle, as, for instance, the circle BEGF in the above figure, of which the centre A is any point in the line CD, and consider this circle as the representative of all space round that point, it follows that the line CD, that is, the part of it BG which passes through the centre and meets the circumference both ways, in B toward the one hand, and in G toward the other, divides the circle into two parts which are exactly equal, so that whatever can be proved as being true of the one of them must be equally true of the other, considered as a magnitude. They are, no doubt, of that description which we have named *symmetrical* magnitudes, that is, magnitudes of the same measure, but differing from each other in position; but then, from the very facts of the centre of the circle being in the straight line separating them and the circumference being everywhere at the same distance from this centre, we have every reason to conclude that they are perfectly equal in every respect, and no reason whatever to entertain even a suspicion that they are not equal.

Simple as it seems when analysed, this is a most important relation between the division of a circle by a straight line passing through its centre, and the division of all space in the plane of that circle; for, it follows almost immediately from this, that in what proportion or ratio soever two lines which meet at any point divide the circumference of a circle, they divide all space round that point, and in the plane of that circle, in exactly the same proportion or ratio. If they contain half the circle, in which case they are in the same straight line passing through the centre, they divide space into two equal parts. If they contain a fourth of the circle between their extremities which meet the circumference, they will also contain a fourth part of space round the point which is the centre of this circle; and as there is no reason from which we can even suspect that the ratio or proportion can be different in any other case, we may receive it as a general, and as nearly as possible a self-evident truth, that, as the entire circumference of any circle described round a point, is the measure or representative of all space round that point in the plane of the circle, so any portion whatever of the circumference of a circle, and any portion whatever of the space round the centre of a circle considered as a point, and contained between the two lines which divide off, or mark off that portion of the circumference, are mutually the measures of each other, and that, therefore, either of them may be used as the representative or expression for the other.

From this it follows, that we have only to apply our arithmetic to the circumference of a circle, in order to be in possession of a standard, or scale, for the measurement of any portion of the space round a point; and as any circle may be taken for this purpose, the scale may be expressed in terms of a circumference, and not in terms of any straight line as a standard, as we require to do in the measuring of lines. There is much advantage even in this, for, as no part of a circle is a straight line, it is easy to see (though, in the mean time, we are not called upon to prove it) that no definite portion whatever of the circumference of any circle can be exactly equal in arithmetical proportion—that is, in a proportion fully and perfectly expressible by any numbers however large, to any straight line whatever.

When, therefore, we speak of circular measures, we speak of them generally, not in terms of a foot, a yard, a mile, or any other measure, but in terms of the whole circumference of which they are parts, and without any regard whatever as to whether this circumference, if we were to try to express it as nearly as possible in terms of straight lines, were equal to the thousandth part of an inch, to ten thousand millions of miles, or to the longest line which imagination could fancy to exist in space.

There is nothing either puzzling or new in this expression of portions of the space round a point by numbers which do not represent the lengths of any straight lines, for we meet with the very same thing in the most common use of arithmetic. If we use the name of any one thing which has existence or meaning, along with the name of a number, we tie the number down, as it were, to that particular kind of thing, so that we cannot, without contradiction and absurdity, regard it as being, in that case, the representative of anything else. Thus, when we say or write 5 MEN, 5 HOURS, or 5 anything else that we can name, we fix the number 5 to the men, the hours, or whatever else is named; and the number, in no one of these cases, expresses any part of the quantity which it expresses in the other. We number men in terms of a man; we number hours in terms of an hour; we number every thing in terms of *one* of that thing;

and therefore we are faithful to our arithmetic when we number, or count, or express (for these words have nearly the same meaning in arithmetic) circular measures in terms of ONE CIRCUMFERENCE.

For this purpose, it is of little consequence what scale we take for the division of the circumference; but it is convenient to have larger and smaller denominations, just as we have in the case of money, of weight, of measure, and of all other things which admit of indefinite division into parts. The entire circumference is considered as divided into 360 equal parts, which are called *degrees*, and the short mark which is used in writing degrees is a small $^{\circ}$ on the right hand of the figures, which may of course be any number not greater than 360°; but more than 360° would have no meaning in a single expression, because it would apply to more than one circumference, which could not, of course, be described round one centre.

Each degree is supposed to be divided into 60 equal parts, which are called *minutes*, and are marked in writing by a small dash over the right; thus, $24'$ is read "twenty-four minutes."

For more minute division, the minute is again subdivided into 60 equal parts, which are called *seconds*, and are marked in writing by two dashes over the right; thus, $25''$ is read "twenty-five seconds."

From these definitions it follows that, as the whole circumference of a circle, or the whole measure of space round a point, is 360°, so half the circumference of a circle, or half the space round a point,—which last means the space on one side of a straight line passing through that point,—is equal to 180°; that one-fourth, or quadrant (which just means a fourth), of the circumference of a circle, or one-half of the space round a point, upon either side of any straight line passing through that point, is equal to 90°; and that, generally, any fraction whatever of the circumference of a circle, or of a space round a point, may be arithmetically expressed by the same fraction of 360°: a sixth will be 60°, a twelfth 30°, and so on for every other fraction; for if the fraction or part which the portion of the circumference, or of the space round the point, is of that circumference, or of that space, and the last is a constant quantity, not in any way affected by the lengths of lines, and the first is equally constant as to the number of degrees, is expressible arithmetically in terms of that circumference, or that space, considered as 1 whole, it is only the multiplying the value of this fraction by 360, to express it in degrees; by 60 again to express it in minutes; and a second time by 60, to express it in seconds.

It may not be amiss to bear in mind the extent of the circumference, or of the space round a point, in each of the three measures, the minutes being found by multiplying 360 by 60, and the seconds by multiplying this product by 60; and from these simple multiplications we have the following results:—

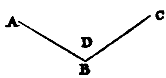
One circumference, or all the space round a point is = 360°, or 21600′, or 1296000″.

In some cases it is convenient to use a greater degree of accuracy than seconds, and the modern way of doing this is by expressing whatever is less than seconds in decimals of a second.

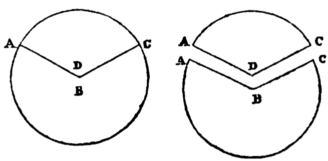
The minutes and seconds which are here alluded to as expressing portions of the circumference of any circle, or of the measure or space round any point, must not be confounded with the minutes and seconds of time which we use as subdivisions of the hours of a day. There is the same number of minutes in the degree as in the hour, and the same number of seconds in the minute in both cases; but the meaning is not exactly the same. There is, however, a sort of shadowy relation between them; for the hours, minutes, and seconds, in our common estimate of time, are subdivisions of the apparent daily motion of the sun round the earth, or of its cause or counterpart—the real rotation of the earth in the opposite direction, it being considered that the motion round is performed uniformly, or always at the same rate, and in the plane of the same circle. Therefore, if we divide 360° by 24, which gives us 15°, we have the number of degrees in the apparent motion of the sun, or of the real motion of the earth during one hour; and thus we are in a condition with comparing the one with the other.

From the definition which has already been given of an angle, it determines the portion of space included between two lines which meet at a point; and therefore all the angles which can be formed by straight lines meeting at any one point, in any one plane whatever, are equal to, and may be expressed by the circumference of a circle, or 360°; and from what has been already said, it follows that all the angles made at one point, on the one side of any straight line, and in the same plane, are together equal to the half of a circumference, or to 180°. It follows also, that if two lines, meeting at the same point in the same plane, make angles exactly equal to 180°, these lines stand in exactly the same relation to space both ways, that is, they are in the same straight line. But if two lines meeting at a point make either more or less than two right angles, they must stand in different relations to space on the different sides of them; yet the sum of the two relations will always be equal to 360°, that is, to twice 180°, and the one relation will be exactly as much less than 180° as the other is greater than the same. Every angle which can be formed by the meeting of two lines, can have the arc, or portion of a circle, which is its measure, or representative, described upon it by a circle of any size, that is, with any radius, or at any distance from the point where the lines meet, as a centre; and, as this portion, or arc of the circumference, is the same part of a circle, that is, of the whole circumference, whether the radius is longer or shorter, it follows that any circle whatever will equally become the measure of the angle, or inclination, formed by two lines; and that an angle does not in any way depend upon the length of the lines which mark it off as part of the space round a point.

If the angle is less than 180°, it is called a *salient* angle, that is, an angle, or corner, with its point projecting outwards; but if it is greater than 180°, it is called a *retroflected* or *re-entering* angle, that is, an angle the point of which is directed inwards, or toward the figure to which it belongs. Thus the angle formed at the point B by the two lines A B and C B in the following figure, is a salient angle, considered with reference to the space on the upper side, or the side D; but it is a re-entering angle considered with regard to the space on the opposite side, that is, on the under side, or the side B.



This will be more apparent if we describe the circle on

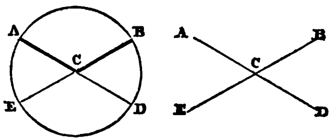


the point B, with any distance for radius, as in the preceding diagrams.

Thus the arc, or portion of the circle from A to C, measured on the upper part of these two figures, is the measure of the salient angle ADC of the upper surface or area contained by the upper portion of the circle, and the two lines AB, CB, while the lower part of the circle from A to C is the measure of the re-entering angle at B, in the lower surface which is contained within this last-mentioned part of the circle and the lines AB, CB. For the better understanding of this, we have repeated the same figure in the second diagram, only the two surfaces are in this repetition placed at a distance from each other, so that the salient angle of the upper one at D, and the re-entering angle of the lower one at B, may be both seen; but the lines AD and CD in the one diagram, and the lines AB and CB in the other, are all equal, and those toward the same side stand in exactly the same relation to each other in these two separate spaces, as they do in the entire circle. From this it is evident that any salient angle, together with the re-entering angle which it becomes when viewed on the other side of the lines, or any re-entering angle, together with its correspondent salient angle, makes an entire circumference of a circle, or an arithmetical sum of 360°.

If either of the lines which form an angle, let that angle be what it may, is produced beyond the point at which they meet, the portion which this produced line, together with the salient angle, makes, must always be equal to the space at a point on one side of a straight line, that is, to 180°, or answering to the half of a circle; and as the original salient angle is not in the least altered by the producing of one or both of the lines, it necessarily follows that, if both are produced, the portions of angular space which they cut off from the re-entering angle, must in all cases be equal to each other; and also that the remainder of the re-entering angle, after both are cut off, must be exactly equal to the original salient angle.

Thus, in the following figures the salient angle formed by the



thick lines A C and B C, where they meet at the point C, is any salient angle whatever, and it is measured by the upper arc of the circle extending from the point A to the point B; so also the angle on the other side of these lines, from the point A downwards by E, and through D again to B, is the measure of the re-entering angle, which, together with the salient angle A C B, includes the whole angular space in any plane around the point C, and of course includes the whole of all possible angles which can be formed at the point C in any one plane. Now if we produce the line A C till it meet the circle in D, it is evident that the two arcs, A B D and A E D on the opposite side of it, are exactly equal to each other, and that each of them is a semicircle, or 180°. In like manner, if the line B C is produced till it meets the circumference in E, it is as evident that the two arcs, E A B and E D B, are each of them equal to a semicircle, or 180°. Therefore the angle B C D, which is cut off from the re-entering angle by the producing of A C, is exactly equal to the angle A C E, cut off from the re-entering angle by producing the line B C; and as either of those equal angles, together with the given salient angle A C B, and also together with the remaining part E C D of the re-entering angle, after the two angles A C E and B C D are cut off, is = a semicircle, or 180°, it follows that the angle E C D must also be equal to the given salient angle A C B.

But AC and BC, meeting each other in the point C, and produced in the same straight lines to D and E, on the other side of the point C, are any two straight lines whatever, situated in the same plane, and crossing each other in any point C. Also, though the circle is the measure of the inclination of the lines, it does not affect the magnitude of the angles, inasmuch as the circumference of the circle does not meet either of the lines at the point C, and therefore can have no possible influence on the angular affection which the lines have in respect to each other at this point. This angular affection, whatever may be its amount, and in the present case we are putting it generally, or supposing the angle ACB to be any salient angle whatever, must depend wholly upon the inclination of the lines themselves, and in no respect upon the circle. Therefore we may dispense with the circle, and view the lines as in the second of the above figures, which gives us the case of any two straight lines whatever, AD and BE, crossing each other at any inclination or angle whatever, in the point C; and we have from it the following conclusions:—

First, if two straight lines cross each other in any point, the angles which are vertically opposite, that is, the angles which have their salient points, or vertices, turned towards each other, must always be exactly equal; as, in the above figures, the angle ACB must be = the angle ECD, and the angle ACE = the angle BCD, whatever may be the magnitude of one of each of these pairs taken singly.

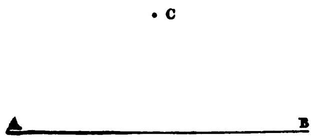
Secondly, that the two angles formed on either side of any straight line, by any other straight line which meets it or crosses it at any point, must always be together equal to 180°. The two angles which are thus formed on one side of a straight line by another straight line meeting it, are called the *supplements* o each other; and as their sum is a constant quantity, being always equal to 180°, it follows that, by how much soever either of them is greater than 90°, or the half of this sum, the other must be exactly as much less than the half of this sum.

Thirdly, as there is nothing concerned in the magnitudes of the angles in this case, except the fact of one straight line meeting another at any point; and also, as the two angles on the same side of the line which is met by the other, are always equal to 180°, it follows conversely, that, if we can show that the two angles which two straight lines meeting another straight line at the same point from opposite sides, are equal to 180°, that is, to half the angular space round a point, then it follows that these two straight lines are a continuation of one and the same straight line.

Fourthly, as lines have no breadth, it follows that the sum of all the angles made at a point on one side of a straight line, whatever may be their number, is, together with those made by the nearest ones and the straight line which they meet, equal to 180°, that is, to half the angular space round a point; and also that all the angles which can possibly be made at one point in the same plane, are together equal to 360°, or the entire angular space round that point, as measured by the circumference of a circle.

The vertically opposite angles which are made by two straight lines intersecting or crossing each other in any point, are symmetrical magnitudes, or rather symmetrical relations, that is, equal indications of the lines, but lying in opposite directions to each other; so that the proof of their equality rests upon the doctrine of the sufficient reason, which is perfectly satisfactory proof in their case, as the lines have no quality but length and direction, and it is the direction only which determines the magnitude of the angle; so that the inclination, or difference of direction in the lines, is not merely a quantity equal to the angle, but the angle itself, whether the lines be or be not continued till they meet, so as that the angle formed by their meeting is apparent to the eye.

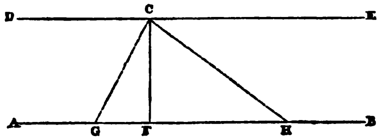
This is a very important consideration in elementary geometry, and one that requires to be thoroughly understood by every body who wishes to be well grounded in mathematical science; and therefore we shall view it in another light, namely, that in which the position of a plane is supposed to be fixed; that is, when three points, not in the same straight line, are supposed to be given in the plane. For this purpose, let A B be any straight whatever, passing through the two points A and B, and continued indefinitely both ways through space, and let C be any point in the same plane with the line A B, but so situated in that plane, as that the line A B, let it be continued ever so far either way, cannot possibly pass through the point C.



Then, it is evident that any line, or number of lines whatever, might be drawn to the point C, in the same plane with the line A B; and that of these lines only one could be parallel to the line A B, or lie in exactly the same direction with it, so as not to meet it, though produced ever so far both ways; and that every other line which could be drawn through the point C, would meet and cut the line A B, either in the direction of A, or in the direction of B, if both the line A B and the line passing through C were continued far enough. How far this might require to be done, would depend on the distance of the point C from A B, and on the inclination of the lines which were drawn through C, taken jointly. We are not, in the mean time, called upon to state numerically the amount of the angular inclination of any line drawn in this manner, in terms of a circular arc. But it is evident that if we suppose *the parallel* line D E, in the figure on next page, to be drawn through the point C, then, as that line is not in any way different in direction from the line A B, that is, has no inclination to it or from it either way, any other line drawn through the point C so as to meet the line A B, and cross it, (and every line except the parallel D E would do this,) then, as the two parallel lines have exactly the same direction, this third line would make, with both of them, the vertically opposite angles exactly equal to each other. Also, if a line were drawn from the point C in the one of the parallels, directly toward the other parallel, that is, exactly bisecting the angular space at C, on the side of D E, nearest to A B, and meeting A B in some point F, then this straight line would also divide into two equal parts the angular space round F, on the side of A B nearest to C D.

Thus, in the following figure A B is any straight line, C any point in the same plane with A B, but apart from it, how far soever it may be continued, and D E is the parallel to A B, drawn through the point C. Then, if we suppose C F drawn from the point C in D E, so as to make the angles D C F, and E C F equal to each other, the angles A F C and B F C must also be equal to each other; and each of the four must also be exactly one-fourth of the angular space round a point, or measurable by 90°, the quarter or quadrant of the circumference of a circle. This being the case, it is farther evident that the two angles on each side of the line which meets the parallels, that is, the angles A F C and D C F, on the one hand, and the angles B F C and

E C F, on the other hand, are together exactly equal; and that the sum of the two, on either hand, is half the angular space round a point, or 180°.



As the very definition or notion which we have of parallel lines, is that of their having exactly the same direction, it necessarily follows that there cannot be drawn through the same point more than one line parallel to the same line; that, for instance, there cannot be drawn through the point C any line parallel to the line A B, except the line D E, for if we were to attempt to draw another parallel, it would have exactly the same direction as D E, and coincide with it, and be, in fact, identically the same; for though lines do not occupy space, yet two distinct lines can no more be supposed to exist in the very same situation, than two solids can occupy at the same time the same identical portion of space.

But as there cannot possibly be two parallels to A B drawn through the point C, so also there cannot be two lines drawn from C towards A B, so as equally to divide the angular space at C, on the side of D E, which is toward A B; and therefore, as one line cannot pass through two points which are not both in its direction, there is only one point F, in the line A B, in which a line drawn from C in the parallel, and dividing the angular space at C into two equal parts, can meet the line A B. This point F is a definite point, if the point C is so; and the line C F, which goes, or is drawn, *right* or *directly* from C to F, has thus a property different from any other line which could be drawn from the point C, in D E, to any other point than F, in the line A B. It makes the angles A F C and B F C, which are called the *adjacent* angles, as the one lies at the side of the other, exactly equal. The line C F is called a *perpendicular* to the line A B; it is the only perpendicular which can be drawn joining that line and the point C; the angles on the opposite side of it, or the adjacent angles, as already explained, A F C and B C F, are called *right angles*, because the line C F is inclined right, or directly toward the line A B, and not more to the one side than to the other side.

A right angle thus means the fourth part of the angular space round a point, or the half of that on one side of a straight line passing through the point; and consequently, the circular measure of a right angle is always the fourth part of a circumference, or 90°; and all right angles, considered merely as expressing angular position, and without any reference to the particular lengths of the lines which form them, are equal.

The two angles which the parallels make with the perpendicular on each side, are both right angles, and therefore the two on each side are together equal to two right angles, or the four are equal together to four right angles, that is, to the whole angular space round a point, or, estimated in circular measure, to 360°, or an entire circumference. If we are to suppose any other line or lines besides the perpendicular C F, to be drawn from the point C, so as to meet the parallel A B, each of such lines would be inclined either to the right hand or to the left. Thus, for instance, the dotted line C G, which meets A B in G, is inclined to your left of the perpendicular C F, and the dotted line C H is inclined to your right of the same; and if we could suppose the line A B to be continued far enough, there might be an indefinite number of lines drawn from C to meet it in all imaginable points, both to the right and left of the point F, in which the perpendicular meets it. It is also evident that each of these lines would be inclined more to the one hand than to the other; that is, that of the two angles which it formed with the line DE at the point C, the one would be greater than a right angle, and the other less: and that, as the sum of these two angles, how much soever they may differ from each other, is still the same measure, that is 180°, or half the angular space round a point, the one of them would be just as much greater than a right angle as the other was less.

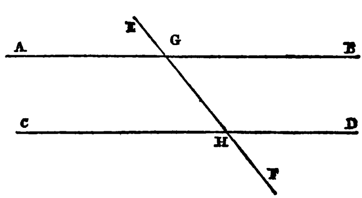
Thus, in the case of CG, the angles DCG and CGA are still equal to two right angles; but DCG is less than a right angle by the angle GCF contained between GC and the perpendicular CF; and AGC is greater than a right angle by the same angle GCF, formed by GC and the perpendicular.

Now if DCG is less than a right angle by the angle GCF, and if AGC is greater than a right angle by the same angle GCF, it follows that the angle GCF is half the difference of the angles DCG and AGC; for $(90^{\circ}+GCF)-(90^{\circ}-GCF)=2\,GCF$.

In like manner, if we examine the angles made on the other hand by the dotted line CH, we find that the angle ECH is less than a right angle by the angle HCF, formed by the line CH and the perpendicular CF, and that the angle CHB is greater than a right angle by the same angle HCF.

Therefore it is apparent that, as the interior angle at the one end of a line which falls upon or crosses two parallel lines, becomes, in all cases, just as much greater than a right angle as the interior angle on the other side is less, the sum of those two angles must, in every possible position of the line which falls upon or crosses the parallels, remain equal to two right angles, because it needs no argument to prove that, if the one of two things always increases at exactly the same rate at which the other diminishes, then the sum of them must be perfectly invariable. This must be true of the sum of the angles on the one side of the line, as well as on the other side; and this puts us in possession of all the facts with regard to every straight line which can fall upon or meet two parallel straight lines, and which, indeed, might be inferred without any reasoning, from the fact of the parallel lines being in exactly the same direction, that is, as having no inclination toward each other, or the least tendency to meet, if produced ever so far.

These facts may be briefly stated thus:—if a straight line falls upon two parallel straight lines, the angles which it makes with them toward the same parts are exactly equal to each other, those which are formed with the same line of the parallels being reversed or symmetrical on the opposite sides of each line, and equal only when vertically opposite, except in the single case in which the line falls at right angles on the parallels, and then all the eight angles which are formed at the two intersections are equal, and each of them is a right angle. In all other cases there are four equal ones, and other four all equal to each other also, but each differing as much from a right angle + or − as the other differs from a right angle − or +. The following figure will show how this part of the subject is usually stated, and which of the angles are equal to each other in every possible position of the line which falls upon the parallels:—



A B and C D are any two parallels upon which any line whatever

EF falls, crossing AB, in the point G and CD, in the point H; and it follows, from what has been said, that all the angles which are vertically opposite at the crossings are equal to each other, namely, AGE, BGH, GHC, and FHD are all equal to each other; also EGB, AGH, GHD, and CHF are all equal to each other; and any one of these together with any one of the former four makes two right angles, so that they are respectively the supplements of each other.

Angles at opposite sides and ends within the parallels are called *alternate angles;* and in our example, they are AGH and GHD, and BGH and GHC. These alternate angles are always equal to each other.

Of the angles on the same sides of the parallels, and also on the same side of the line falling on them, the one is called the exterior, as, for example, AGE, and the other the interior, and opposite, as, for instance, CHG. The exterior angle and the interior one opposite are also equal to each other in every possible case.

We need hardly add that the four exterior, and also the four interior angles, are always together equal to four right angles, and that the two interior ones on the one side of the line which meets the parallels are always equal, not only in their sum but in the individual angles, only that the equal ones are turned opposite ways, or placed symmetrically at opposite ends of the crossing line, just in the same manner as the vertically opposite and equal angles made by one line crossing another are placed; only in the case of the parallels the points of the equal interior angles are placed opposite to each other, being vertically opposite to the equal exterior ones.

If there are three or more parallels in the same plane it is easy to see that if two of them are each parallel to a third one they must be parallel to each other; and that whatever can be said of an intersection of a straight line with any one of them can be equally said of the same straight line with regard to the intersection of all the rest.

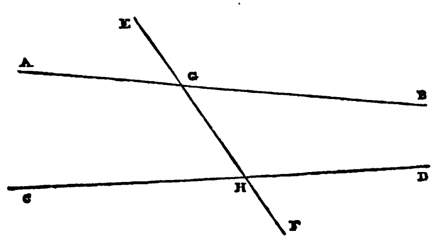
In the case of parallel lines, and lines falling on them, whatever holds true directly must also hold true conversely; so that if a line falling upon or crossing other two lines makes the alternate angles equal, the exterior angle equal to the interior and opposite on the same side, or the interior angles on the same side together equal to two right angles, we may always conclude that the two straight lines, which answer any of these conditions with the third line, are parallel.

Let us now consider the case of two lines in the same plane which are not parallel to each other; and here, upon looking back to the diagram on page 246, and imagining all possible lines to be drawn through the point C at every possible slope or inclination from the perpendicular CF to the parallel CD the one way, and CE the other, we include every possible case; for the perpendicular CF is equally inclined to the parallels, or makes exactly right angles with them both ways; and we have already shown that any line except the perpendicular must make as much more than a right angle the one way as it makes less than a right angle the other way.

Thus the perpendicular is one limit; and though we do not require the establishment of that fact, in the mean time it is evident that the perpendicular is the shortest line which can possibly be drawn from the one of two parallels to meet the other, produced if necessary, and that all perpendiculars which can possibly be drawn from one of the same parallel lines to the other must be exactly equal. In the mean time, however, we have to deal with the angles only, and not with the lengths of the lines; and, as the space round a point on one side of a line is always two right angles, which is the same quantity whether we divide it into two equal parts or two unequal ones, it is plain that the perpendicular divides it equally, that is, into 90° on the one side, and 90° on the other.

The 90° in the case of the perpendicular, may be considered as wholly an inclination of that line *toward* the line which it meets; but in every other case the same 90° may be considered as divided into an inclination toward the line, that is, toward the perpendicular and an inclination from it, till we come to the parallel, in which the whole 90° is an inclination from the perpendicular. From this we may get another definition of parallel lines, which would answer our purpose nearly as well as that which we have chosen, namely, that they are lines in the same plane, to one of which, if a third line also in the same plane is perpendicular, it must be perpendicular to the other.

Let us now consider lines in the same plane which are not parallel, but which have some inclination to each other, so that they would meet if produced far enough; and it is not necessary to our present purpose that they should actually meet, but only that they should have an inclination. From what has been already said, it will be apparent, that how much soever they are inclined toward each other in the one direction, they must be inclined exactly as much from each other in the opposite direction. Let A B and C D be two such lines,



inclined toward each other in the direction of B and D, and from each other in the direction of A and C, let any line, as E F, fall upon them in the points G and H, and let us consider how their relations to this line will differ from the relations of two parallels. It is evident that in this case, as well as in every other case in which two lines in the same plane intersect or cross each other, the vertically opposite angles must be equal to each other at each crossing, and also that at each crossing the two adjacent angles on the same side of the line E F must be equal to two right angles, whether we take them at the point G or at the point H, and whether we take them in the direction toward D and B, or in that toward A and C. But when we take the two interior angles on the opposite sides of E F, that is to say, the angles G H D and H G B on the side toward B and D, and the angles A G H, G H C on the side toward A and C, we perceive that in no case, unless where A B is parallel to C D, can these two interior angles on each side be together equal to two right angles, nor can the two on the one side be together equal to the two on the other, though the sum of all the four must still be equal to four right angles. Therefore, the two interior angles on the one side must be greater than two right angles, or 180°, and the two angles on the other side must be just as much less than 180° as the other two are greater. Now, as the sum of each of those pairs of angles would have been exactly equal to two right angles if A B had been parallel to C D; it follows that the difference between the sums of these two angles respectively and two right angles is the measure of the inclination of the lines A B and C D, that is, it is the measure of the angle which they would form if produced till they met.

But we have data for finding this angle, without any regard whatever to the distance to which the lines would require to be produced in order to meet; for the angles on one side of E F, interior of A B and C D, are together exactly the measure of this angle more than two right angles, or 180°, and the sum of those on the other side is exactly the same measure less than two right angles, or 180°.

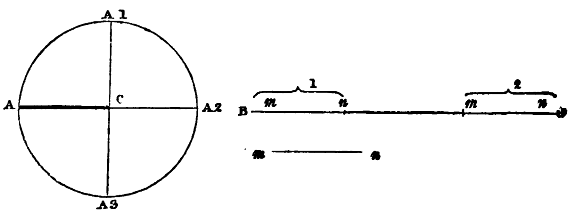
The smaller interior angles evidently lie on the side toward B and D, and the larger angles on that toward A and C, because A B and C D are inclined *to* each other in the first of these directions and *from* each other in the second. Therefore, if we take away the two interior angles, B G H and G H D, from two right angles, that is, from 180°, or half the angular measure round a point, the remainder is the angle formed by the two lines at the point where they would meet. But the angles G H D + D H F are exactly two right angles; and G H D + B G H are less than two right angles. Take away the angle G H D, which is the same in both, and it follows, calling the angle in which the lines would meet x, that D H F, the exterior angle, — B G H the interior and opposite angle, must be $= x$, and that the exterior angle must be $=$ B G H $+ x$.

Also, calling B G H simply G, G H D simply H, and bearing in mind that x is the angle at which the lines G B and H D meet, and that G x H would be a triangle, we have the sum of the angles of *any* triangle, without any reference to the lengths of the sides, namely, the sum of those angles, that is, the sum of the angles of any plane triangle whatever, is a constant quantity equal to two right angles, or 180°, and that G x H would be a triangle.

Therefore, if we know two angles of a triangle, we can at once find the third by substracting the sum of the known ones from 180°, and if we know one angle we can find the sum of the other two by substracting the known one from 180°.

In the course of the remarks by which we have been led to this conclusion, that the three angles of a triangle are equal to two right angles, or to 180°, or half a circumference in circular measure, and which is a truth of so much importance that it leads to the determination of the shapes of all straight-lined plane figures with very little investigation, we have necessarily supposed some things to be done, of the way of doing which we are still ignorant; but they all belong to that class of operations which are geometrically possible; and we must first assume the possibility before we perform the operation.

This truth admits of so many applications, and deserves to be so well understood, that we shall show how it may be more simply, or at all events more briefly arrived at, by a method different from what we have stated, though founded upon exactly the same principles. We shall first give a short account of the means employed, and then of the proof, as apparent on the application of those means. Motion, of two kinds, first circular or angular motion, exactly the same as that by which every circle must be described; and secondly, rectilineal or straightforward motion, such as must be used whenever a straight line is drawn: consequently, both are in the legitimate province of geometry, and in fact included to the full extent in the postulates to Euclid's Elements. For illustration, let us introduce the following diagrams, the left hand one for the circular motion, and the right hand one for the rectilineal.



In the left hand figure, or the circle, we shall suppose the point C to be any point in a plane, and the thick line A C toward the left to be any radius given for the purpose of describing a circle round the point A. From what we formerly said (and indeed it hardly requires any pointing out) the only way in which the circle can possibly be described is by turning the radius round upon the point c, and the radius will mark with its extremity the circumference, sweep over the whole surface within the circumference, and in the course of the revolution point in every possibly angular direction from the centre and on the plane.

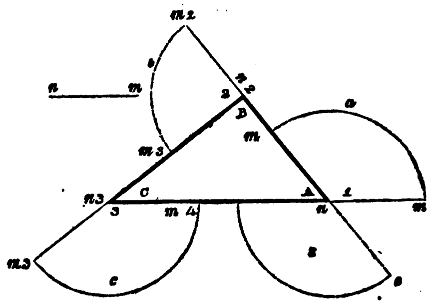
When it has swept over the first quadrant and first right angle, so as to have the position C A 1, it is perpendicular to its first position, or stands to it at an angle of 90°; and it has come gradually to this angle, from 0, or no inclination, at its first position A C. Let us next suppose it carried round another quadrant, or to the position C A 2, and it will evidently have described another 90°, or 180° in all from the beginning. Its position is now reversed as to what it was at starting, the end which was seen to the left hand being now to the right, and it is now in a straight line with the first position. In passing the two remaining quadrants there is merely a reversal of what has been already passed over; and we have as clear a demonstration as can be obtained, that when it has arrived at exactly the first position AC, it will have passed over four right angles, or the entire circumference or space round a point.

In the case of a single circle, that is, a circle described from one centre, this would never be disputed; but we wish to prove that a radius can be proved to have passed over angular space exactly equal to a circumference when it is turned round upon several points; and this requires a little more showing.

On examining the second of the above figures it will not be doubted that if there is a longer line BG, and a shorter one *m n*, that the shorter one can be applied to the longer in the position toward the extremity B, and marked off by the dots at *m* and *n* numbered 1. This is in fact the first step in all measuring; and we may, if we please, call BG a board, and *mn* a carpenter's rule to measure it with. Now, the measuring would be of no use if the carpenter could not carry his rule straight forward along the board, and also stop with it whenever he chose; for instance, at the position marked *mn* toward the end G of the line, and numbered 2. It is plain, that as long as the line *mn* is carried straight forward along the line BG there cannot be the smallest change in its angular direction; and that, so far as the carrying is concerned, there would be no more change in the angular direction whatever number of times it were carried along. By putting those operations together we are enabled very clearly to prove the truth of the following

Theorem.—All the exterior angles of any rectilineal figure having all its angles salient angles, are together equal to four right angles. Let us take a particular case and show this.

We shall first take the case of a triangle as the simplest figure.

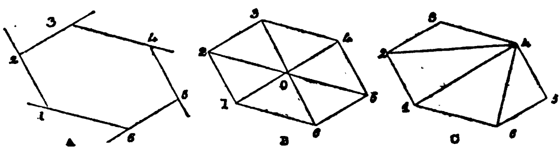


Let ABC be any triangle of which the interior angles are respectively at A, at B, and at C; and let this triangle, which is marked in thicker lines than the rest, be regarded as the datum or thing given. Produce the sides in the same straight lines in the directions *m* 1, *m* 2, and *m* 3; and the angles 1, 2, and 3 are the exterior angles, each making together with its adjacent interior angle two right angles, or 180°. It is evident that it is of no consequence at which extremities the sides of the triangle are produced, so that each of them is produced at one extremity; for the angle formed by B A, produced to *o*, is vertically opposite to the angle 1, and consequently must be equal to it, and it would be the same at the other angles. Let there now be taken the straight line *m n*, and let it be applied to the produced part CA, that is, let it have the position *m n*. Turn it round on the angle A as a centre, and it will describe the arc *a* and the angular space 1, and coincide with the part of AB extending from *n* to *m*. Move it straight forward along AB and its continuation, till it has arrived at the position *n* 2 *m* 2. Turn it round on the angle B as a centre till it coincides with the part of CB marked *n* 2 *m*3, and it will describe the arc *b* and the angular space 2. Move it straight forward along BC and its continuation, till it has the position *n* 3 *m* 3. Turn it round on the angle C till it coincide with the part *n* 3 *m* 4 of AC, and it will have described the arc *c* and the angular space 3. Move it along AC and its continuation till *m* 4 arrives at the point *m*, and *n* 3 at the point *n*, and it will have exactly its original position, or have been turned over angular spaces amounting altogether to four right angles, and will have described arcs of circles amounting altogether to an entire circumference. But it has been moved round the exterior angles of the triangle only, and therefore all the exterior angles of any triangle are together equal to four right angles. The exterior and interior altogether amount to six right angles; take away four for the exterior; and the sum of the three interior angles is equal to two right angles, as before.

The measure of the three interior angles might be shown directly to be equal to two right angles, by applying the line to one of the sides, turning it round on one angle, moving along again and turning on the next, moving along again and turning on the third angle, after which the line would be found upon the opposite end of the side to which it was first applied, with its extremities reversed, which is the position of a radius after having described a semicircle.

The general fact of every possible straight-lined figure which has all its angles salient, having the sum of the supplements of those angles always equal to four right angles, might be proved by the very same means which we have used in the case of the triangle; for, if a radius line is carried along all the sides and turned round on all the exterior angles the *same* way, it must in every case come to its first position when it has been carried round the whole figure.

This may be seen by mere inspection of the first of the following figures, or by any other figure which could be drawn, if bounded by straight lines, and having only salient angles.



Then it follows that, as the exterior angles are always equal to four right angles, all the interior angles of every rectilineal figure are equal to twice as many right angles wanting four, as the figure has sides.

It is only the sum of the angles that is known, except when they are all equal, and then each angle is the sum divided by the number.

Using the expression *rt. ∠les*, for "right angles," we shall state the angles of a few of the regular or equal-sided plane figures, with the names:—

SIDES.	NAME.	WHOLE ANGLES.	EACH ANGLE.
3	*Triangle*	6 rt.∠les—4 rt.∠les = 540°—360° = 180°	60°
4	*Square*	8 rt.∠les—4 rt.∠les = 720°—360° = 360	90°
5	*Pentagon*	10 rt.∠les—4 rt.∠les = 900°—360° = 540	108°
6	*Hexagon*	12 rt.∠les—4 rt.∠les = 1080°—360° = 720°	120°
7	*Heptagon*	14 rt.∠les—4 rt.∠les = 1260°—360° = 900°	$128\frac{4}{7}$°
8	*Octagon*	16 rt.∠les—4 rt.∠les = 1440°—360° = 1080°	135°
9	*Enneagon*	18 rt.∠les—4 rt.∠les = 1620°—360° = 1260°	140°
10	*Decagon*	20 rt.∠les—4 rt.∠les = 1800°—360° = 1440°	144°

Comparing the numbers in this last column, we find the following series of differences, taken in the order of the table:— 30°, 18°, 12°, $8\frac{4}{7}$°, $6\frac{3}{7}$°, 5°, 4°; and we may have occasion to revert to these differences afterwards.

It may not, however, be amiss to glance back at the figures B and C in the cut on the preceding page, because, from the simple inspection of them we may see how the sum of all the angles of any plane rectilineal figure can be found by means of the triangle, and also that any straight-lined figure whatever may be reduced to triangles, and expressed by them. There are two ways of doing this, one of which may be illustrated by each figure.

Take, as in figure B, any point within the figure, and draw straight lines from this point to all the angles and the figure is divided into as many triangles as there are sides; each triangle has one angle at the centre, and all the angles thus are equal to four right angles, for, whatever may be their number, they are all the angles round a point, and they are no more. But the sum of the angles of the triangles, exclusive of those round this point, is equal to the sum of all the angles of the figure; therefore the sum of the angles of this figure is equal to twice as many right angles, wanting four, as the figure has sides.

Again, as in figure c, draw as many lines, from any one angle to the other angles, as will divide the figure into triangles, and as in this case two sides of the figure are required for each of two of the triangles, there are two triangles fewer than the figure has sides; but the angles of all the triangles are equal to all the angles of the figure; therefore, again, all the angles of the figure are equal to twice as many right angles wanting four as the figure has sides.

We are now in possession of all the more important elementary cases in which angles can be shown to be equal from the consideration of the direction of lines only, and without any reference to the lengths of lines or the measures of surfaces; and as these truths, *after the way of arriving at them is known*, are worth remembering, we shall repeat them in brief, it being understood that, in them, all the lines are in the same plane:—

1. When lines cross each other, the vertically opposite angles are equal.

2. When one line meets another on one side, the two angles, if equal to each other, are both right angles; and they are together equal to two right angles whether they are equal to each other or not. When they are unequal each is as much greater than a right angle as the other is less, and they are the supplements of each other.

3. Lines which meet another straight line on opposite sides at the same point, and which make the two adjacent angles equal to two right angles, lie in the same direction, or are in the same straight line.

4. Lines which cross or fall upon parallel lines, make the

exterior angle equal to the interior and opposite, the alternate angles equal, and the two interior angles on the same side of the line which falls on the parallels, equal to two right angles.

5. If each of any number of lines in the same plane can be shown to be parallel to the same line, they are all respectively parallel to each other, and any line crossing them makes equal angles with them.

6. If, when a straight line falls upon or crosses two other straight lines, in the same plane, it makes the two interior angles on the one side less than two right angles, it must make the two on the other side just as much greater than two right angles, and the two lines on which the third line falls must meet on that side of the third line upon which the two interior angles are together less than two right angles, that is to say, if the two lines are produced far enough.

The angle which two such lines would make with each other, if produced till they met, is the measure of their inclination *to* each other in that direction in which they would meet, and *from* each other in the opposite direction; and as the inclination of the lines is the same, whether they are produced till they meet or not, the angle which expresses this inclination is always a known quantity; that is, a quantity which we have sufficient data for finding: namely, the difference between the sum of the two interior angles and two right angles, or 180°.

7. The exterior and the interior and adjacent angles are always, together, equal to two right angles, and the three angles of a triangle, taken altogether, are also always equal to two right angles; therefore, the exterior angle of a triangle, made by producing one of the sides, is always equal to the sum of the two interior and opposite angles of the triangle; consequently if this exterior angle and one of the interior and opposite ones

are known in circular measure, the other can be found at once by substracting the known one from the exterior angle.

8. All the exterior angles of any straight-lined figure, having all its angles salient or pointing outwards, are together equal to four right angles, and all the interior angles are equal to twice as many right angles wanting four, as the figure has sides. Hence the four angles of every four-sided figure, whatever are the lengths of its sides, will always exactly coincide with or cover the space round a point, if all their vertices are brought into contact at that point.

All the principles stated in this section are complete and simple; that is, they depend upon only one condition; therefore the truth of the converse or opposite of each of them follows as a matter of course. This is a very important general maxim, though it is one which, probably on account of its great simplicity, is rarely stated; but when it is admitted as general, it saves a great deal of unnecessary labour, and gets rid of a good deal of that perplexity which beginners feel in studying the elements of geometry. Before we apply it, however, we must be sure that we are in possession of all the conditions, each as single and simple; because if any one of them is compound, and we do not know its composition, the general maxim will not apply. A very simple case will show this:—that the sum of 2 and 1 is 3, is an absolute truth; but the converse—namely, 3 is the sum of 2 and 1, is not generally or absolutely true; for 3 is the sum of 1, 1, and 1, and also of an indefinite variety of other numbers.

That lines which are parallel, or have no inclination to or from each other, never can meet or cross one another in any point, is also a simple and general truth; and though the converse is not so palpable to our common understanding, yet it is equally true that every two lines which are not parallel must meet and cross each other if produced far enough,—it being understood that the lines which are not parallel are in the same plane, or that neither of them is affected by a line crossing it at right angles, and thus as it were tying it down to a different plane from that of the other line. It is this possibility of the two lines which are not parallel to each other being situated in parallel planes, and thus in this particular case not meeting, which renders the converse "lines which are not parallel must meet each other" not absolutely true. When, however, the single condition of being in the same plane is added, the truth becomes as absolute as in the other case; and this is the only consideration of the position of lines which occurs in plane geometry. Parallel lines may always be considered as in the same plane, because it is easy to imagine a plane to be made to pass through any two parallels whatever.

SECTION XIII.

DOCTRINE OF PROPORTION.

The doctrine of proportion, taken in its most extensive sense, is one of the most important, not only in the mathematical sciences, but in every science and every department of human knowledge. It is the general doctrine of the *relations* or *ratios* of all quantities and all subjects, whether existences, qualities, actions, or thoughts, whereof the comparison can be a source of knowledge. In the general sense of the term, it is closely allied to, and indeed nearly identical with, that principle which decides for us in all our judgments, and regulates us in all our actions,—namely, that "in circumstances which are exactly alike, like results may be predicted" with all the certainty which we can possess in any prediction.

When, however, we take only the mathematical view of this doctrine of proportion, it of course extends no further than mathematical subjects extend; but still when we thoroughly understand it in this simple and elementary point of view, we are masters of it as a general instrument of knowledge, inasmuch as the method of its application is the same in the most complicated cases as in those which are of the most simple and elementary nature.

The doctrine of proportion may be considered as in principle wholly a matter of arithmetic, though there are many cases in which even a relation that we understand perfectly in its nature, cannot be at all expressed in numbers; and there are others which cannot be expressed accurately in numbers, though we may approximate their truth to any degree that may be required in any one practical case. Thus, for instance, there is a relation between the state of the season —the Spring for instance, and the degree of development in the vegetable tribes; but there are no means by which we can state in two numbers the cause as operating in the season, and the effect as produced in the growing world. In like manner, the number 10, and the number 3, are both expressible, and in fact expressed in terms of the number 1; but if we were to attempt to get rid of this medium of expression, and try to express 3 in terms 10, there are no means by which this simple expression can be accomplished with perfect accuracy. The decimal ·3 is the first step of the expression, but the relation cannot be accurately expressed unless this ·3 is repeated without end.

These are simple instances, but they show us upon what principle we must deal with more complicated ones, so as to get a relation between each and some third subject, when there is no relation expressible between the two subjects which we wish more immediately to compare. Thus, for instance, Farmer Gubbins has a load of excellent meadow hay over and above what is required for the supply of the horses employed on his farm; and Dame Gubbins and her daughters are in want of a certain web of printed cotton which lies in the shop of Mr. Tape the draper, in order that they may appear at church in seemly guise. Mr. Tape does not want the hay, and therefore there can be no possibility of instituting a ratio, so as to obtain an exchange, until recourse is had to a third element; and Gubbins first turns his hay into cash at the market-price, and then measures that cash against the web of printed cotton, according to the price set upon the latter. In this manner, though the commodity which we call cash, directly serves the necessities of nobody, yet it brings all the people into an exchanging condition with each other, so that each sells what he has to dispose of, and procures what he wants, and the whole are far better served than if time were wasted in vain attempts at finding relations among quantities which have no quality in common. This is the grand practical illustration; and it is this which renders the mathematical or general doctrines of proportion so very valuable.

The word *proportion* literally means "for or as a part," that is, that inquiry shall be made as to what part, expressible by a number or by a fraction, the one of two quantities is of the other; and *ratio* means "the reason," that is, the inference or conclusion drawn from two quantities, which we can make available in acquiring a knowledge of other quantities.

There are two distinctions of relation, according to the object of the inquiry which we make respecting the two quantities compared. The one alludes only to the difference, without any regard to the total magnitude or value of the quantities; and this is called *arithmetical* ratio or relation: the other, to the whole value of the one as compared with the whole value of the other; and this is called *geometrical ratio* or relation. These names are not correct, because both the one and the other may apply to quantities which are strictly arithmetical, strictly geometrical, or neither the one nor the other; and therefore the less that such names are used the better.

In those cases we do not, generally speaking, include quantities which are equal to each other, because from the comparison of such quantities no useful inferences can be drawn: consequently, the useful cases are those in which there is a less quantity and a greater one, and the difference between the arithmetical proportion and the geometrical one may be said to consist chiefly in this, that, in the arithmetical proportion, the ratio is a difference which, added to the less quantity, makes the greater; whereas the geometrical proportion has the ratio a multiplier, which, applied to the smaller quantity, in the common way of multiplication, produces the greater. Thus, if we take any quantity as a beginning, and add to it any equal quantity of the same kind by successive additions, the results which we obtain will be a series of quantities, of which any two following each other in order, how far soever the series may be extended, will have the same arithmetical proportion to each other. So also, if we take any quantity and multiply it any number of times by the same multiplier, the series of products will be quantities of which any two immediately following each other will have the same geometrical proportion. The common difference which is added in the case of the arithmetical proportion is called the arithmetical ratio; and the common multiplier which is used in the case of the geometrical proportion, is called the geometrical ratio. It is obvious that the arithmetical ratio is the quantity which, in the case of two terms of the series, the one of which immediately follows the other, would, by being added to the less, make a sum equal to the greater, or by being substracted from the greater, would leave a remainder equal to the less. It is equally obvious that the geometrical ratio is a number which, applied as a multiplier to the less of the two quantities immediately following each other, would produce a product equal to the greater, or which, applied to the greater, would yield a quotient equal to the less.

When quantities proceed by an equal ratio in either of these ways, the series of terms is said to form an *arithmetical progression*, by the successive addition of equal differences; and a *geometrical progression*, in the case of successive multiplication by equal multipliers.

But it is perfectly evident that, in the case of any number of quantities of either of those kinds, we may begin either with the greater or the less of the series. If in an arithmetical progression we begin with the greater of the series, the additions will be changed into substractions; and if we begin with the greater in a geometrical progression, the multipliers will be changed to divisors. The changing of a difference to be added to a difference to be substracted is, in expression, nothing more than changing the sign + to the sign − : and the changing of a multiplier to a divisor is nothing more than turning the multiplier into the denominator of a fraction, which has 1 for its numerator, and multiplying by this fraction, is exactly the same thing as dividing by its denominator.

When the difference in an arithmetical progression has the sign +, the progression is an ascending one; and when the difference has the sign −, the progression is a descending one.

Also in geometrical progressions, if the ratio is any number greater than 1, the progression is ascending; but if it is any fractional number less than 1, the progression is descending.

As has been shown in a former section, the substraction of a real or positive quantity is exactly the same as the addition of the same quantity with the negative sign; and as division by any quantity is the same as multiplication by the reciprocal of that quantity—that is, by the number 1 divided by the quantity—we get this general definition which is equally applicable to all progressions, whether ascending or descending.

An arithmetical progression is a series produced by successive additions to a first term; *a geometrical progression* is a series produced by successive multiplications of a first term.

It is not necessary that *all the* additions in the case of the arithmetical progression, or all the multipliers in the case of the geometrical ones, should be equal to each other. They may vary according to any law, and hence there is no assignable limit to the number of such series, and the doctrine of series becomes a very extensive, and in many instances a very intricate one.

The general principle is very simple, however, and as nearly self-evident as possible. It is this:—

In any arithmetical series the difference between the first and the last term is the sum of all the partial differences taken as the terms follow each other. If the additions are all positive, then the last term will exceed the first term by their sum; if they are all negative, the last term will be less than the first term by their sum; and if some of the additions are +, and others −, the difference of these will be the difference between the first and last terms; it will be + if the value of the + additions is the greater, and —, if the value of the — additions is the greater.

In a geometrical series, the difference between the first and last terms, as compared with each other, is, that the last term consists of the first one successively multiplied by all the multipliers or ratios of the terms taken in their order; and that if some of those multipliers are reciprocals, the value of the continual product of the whole is the quotient arising from dividing the product of all the multipliers, by the product of the denominators of all the reciprocals.

These general principles apply to every kind of differences, whether they arise from addition or substraction, or from multiplication or division; and also whether they have the sign + or the sign —, only the introduction of a ratio with the sign — into a series must, from what was mentioned respecting the signs in multiplication, change the signs of all the terms which follow it, from + to —, or from — to +, until another negative ratio come in, and that one will change the signs of the terms back to what they were before being affected by the former —.

The shortest geometrical series that we can possibly have is one consisting of two terms, a first or antecedent, and a last or consequent; and as this is the very simplest case, it is the best one in which to acquire an accurate knowledge of the fundamental doctrine of ratio. The two terms may be numbers of any amount, or quantities of any kind, or they may be qualities or relations which have no separate existence; and thus the doctrine can be applied to every possible subject of thought. But there is one condition indispensable to the simplicity of a ratio in every imaginable case; and this condition is, that the two terms of the ratio shall be quantities of the same kind, and in exactly the same circumstances. They must be such in fact, that whatever we can say or declare of the one of them, we can also say or declare of the other, excepting the single circumstance which depends upon the ratio; and if we can do this and at the same time know the ratio, it is perfectly evident that our knowledge of the one term is as complete as it is of the other.

This is the elementary consideration, the neglect of which very generally renders the doctrine of ratios most perplexing, and not unfrequently altogether incomprehensible to beginners; because, if we do not take care to limit the knowledge which we seek from the ratio to that which the ratio can afford us, we evidently seek the remaining knowledge where it is not to be found, and thus even the portion which is attainable is inadequate to its purpose, and therefore useless.

Here it may be very desirable for the reader, who reads these pages as a student, to turn back and re-peruse Section VI., and the two parts of Section VII., because many of the points explained in them bear immediately upon the doctrine of ratios.

The general expression for a ratio is $a : b$, in which a and b stand for any two quantities whatsoever that admit of comparison; that is, which are of the same kind and in the same circumstances. The sign (:), as will be seen by looking back to page 83, is one part of the sign used for division,—namely, the sign ($\div$); but as the sign is simpler in the case of the expression of a ratio, so the signification is also more ample. $a \div b$ is read "a divided by b," which expresses the quotient; and $a : b$ is read "a to b," which means merely the relation, or the difference in that single respect in which they are not perfectly alike to our understanding. It is true that when we come to apply ratios in practice, we must have our quantities expressed in numbers, and find the measure of the ratio by division; but is it only in particular cases—namely, those in which we can actually determine the quantities—that we can proceed to this arithmetically; and there is a countless number of cases, by far the majority of those which enter into our reasonings and judgments, to which arithmetic cannot be accurately or even at all applied; and we must frame our general conception of ratio so as to include these.

When we come to the arithmetical application, the second term b is the divisor, and therefore the standard in terms of which the dividend a is measured so as to find the quotient or measure of the ratio as a single expression. This of course applies only to ratios which can be stated arithmetically; but still it is evident that the imperfection does not consist in any puzzle about the nature of the ratio itself, but only in our not being able to measure the two quantities, so as to express their values in terms of the same standard. Hence, in every case where a comparison of quantities of the same kind can be made, we have the same understanding of the general nature of the ratio whether we are able to state it in numbers or not; and this consciousness of the existence of a ratio is so immediate a perception of the human mind, that it is the best as well as the shortest definition which we can have of quantities of the same kind.

When our general quantities $a : b$ are such that we cannot express them by numbers, we have the most simple case of ratio; and there can be only three varieties:—first, a may be equal to b, or $a = b$; secondly, a may be greater than b; or $a \mathrel{7} b$; thirdly, a may be less than b, or $a \angle b$. The first of these is a ratio of equality, and it of course admits of no variation, and requires the use of no arithmetic.

The second is a ratio of majority, and the third a ratio of minority; and in each of those cases the question of difference or of quotient may be put. If it is a question of mere difference, "how much greater or how much less," it is evident that the value of the smaller quantity, and of as much of the larger as is equal to the smaller, does not enter into the estimate of the difference. This is the simple matter of the ratio of two terms in an arithmetical progression.

But if the question be the comparison of the whole value of the one quantity with the whole value of the other, we must apply our common means of measurement, by dividing the quantity to be measured by the standard of measure; so that when we have got the values of a and b arithmetically expressed in the same denomination, we have only to apply our common arithmetical division; and the quotient is the ratio in terms of the divisor considered as one whole. Thus, in $a : b$, $\frac{a}{b} : 1$ is the expression of the ratio.

It will at once be perceived that the obtaining of this expression is nothing more than dividing the two quantities a and b, separately, by the same quantity b, and that the expression of the ratio as a quotient in those cases in which it can be expressed is nothing more than reducing the two terms of the ratio in such a manner as that the second term b shall be 1.

If we suppose the case to be one which admits of arithmetical expression and complete division, so that we have the quotient of $\frac{a}{b} = c$; then, according to what was stated when treating of division, $a = b\,c$.

EQUALITY OF RATIOS.

This is one of the most important points in the whole doctrine of proportion; and, as is mentioned at page 216, it is one, with Euclid's definition, of which beginners always feel a great deal of difficulty; and as that definition is generally the first and frequently the only one which they get, the doctrine is rendered little better than useless to all except those who, by giving themselves up, for a long time at least, to the study of mathematics, get at it, as they do at most other matters, the best way they can.

The definition is in these words :—"If there be four magnitudes, and if any equi-multiples whatsoever be taken of the first and second, and any equi-multiples whatsoever of the third and fourth; and if, according as the multiple of the first is greater than the multiple of the second, equal to it, or less, the multiple of the third is also greater than the multiple of the fourth, equal to it, or less; then the first of the magnitudes is said to have to the second the same ratio that the third has to the fourth."

The meaning of this is perhaps rendered a little clearer when it is stated algebraically in precisely the same terms. Thus:—

Let A, B, C, D, be any four magnitudes; that is, any quantities, numbers, measures, or considerations whatsoever; and let m and n be any two numbers whatsoever, that is, let each of them represent in every possible case every possible number, whether it be or be not expressible arithmetically, only each letter must in any one case express the same number, as applied to each of the two magnitudes which it is understood to multiply; and if

$$m\,\mathrm{A} \gtreqless n\,\mathrm{B},\ m\,\mathrm{C} \gtreqless n\,\mathrm{D};$$

then

$$\mathrm{A} : \mathrm{B} = \mathrm{C} : \mathrm{D}.$$

This definition is perfectly correct and general, including all ratios, whether they admit of being expressed by numbers or not; and when once understood, it is not only satisfactory, but very simple. There is, however, considerable difficulty in understanding it, and this difficulty lies in the ambiguous meaning of the words "any equi-multiples whatsoever." Indeed the great ambiguity lies in the word "any," which in the definition means both "any one, and every one;"—that is to say, that the numbers *m* and *n*, which are used as multipliers, shall be any two numbers whatever, in any particular case; and that they shall be applied in all imaginable cases. Now this double meaning, though a beautiful one when understood, is difficult to understand, and instances of it, on far simpler subjects, have sometimes perplexed those who were past the common years of pupilage. Thus in the following line of Gray's matchless Elegy—

Full many a flower is born to blush unseen;

though the poet at once conjures up the whole wild flowers of the desert, and makes each tell in its individual beauty upon the mind of the reader, yet the small dealers in grammar have set this line down as a positive fracture of Priscian's head.

We have mentioned the circumstance of the difficulty which Euclid's definition contains, and pointed out the particular expression in which it consists, chiefly with the view of calling the reader's attention to this definition; because, when it is fully understood, the doctrine of proportion, as expounded in the Fifth Book of the *Elements*, is a very beautiful specimen of geometrical reasoning; and we may perhaps venture to hope that the slight explanation which we have given may assist in getting the better of the difficulty. For our own purpose, and taking, as we have taken, all the elementary branches of mathematics along with us, the defining of the equality of ratios is a very simple matter, and at the same time not less general than that of Euclid.

The ambiguity which, in his definition, is thrown upon the making of a case individual and universal at the same time, is, according to the plan which we shall follow, reduced to incommensurable numbers, or those which cannot be exactly expressed in terms of the arithmetical scale, and to incommensurable quantity, the one of which cannot be exactly expressed in terms of the other. But as in any practical instance their values can be approximated to any degree of accuracy that may be required to a far greater degree indeed than our instruments and our eyes will carry us in actual measurement, it answers every purpose. The principle in this case—and it is self-evident—is, that quantities which have equal measures are themselves equal. This, as we remarked in a former section, is one of our simple and original perceptions of equality, which cannot be rendered more evident by any explanation. From this it immediately follows that, if

$$\frac{a}{b} = \frac{c}{d}, \text{ then } a : b = c : d;$$

that is, if the quotient of the first divided by the second is equal to the quotient of the third divided by the fourth, then the first is to the second as the third is to the fourth, or the ratio of the first and second is equal to the ratio of the third and fourth; because these quotients are the measures of the ratios; and, as they are equal to the ratios, these ratios themselves must be equal.

Stated in this way, the principle applies to all commensurable quantities of which the quotients can be expressed in numbers of any kind; but as it does not apply so well to cases where the quotients cannot be expressed in numbers, it is advisable to make a little alteration in the form. Now, as the ratios are equal, and the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are also equal, the quotient arising from the division of either of these fractions by the other must be $=1$; it follows that the quantities which are divisor and dividend in the finding of this quotient must be equal to each other. To divide one fraction by another, as shown in a former section, is to invert the divisor, and multiply. Hence, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}\times\frac{d}{c}=\frac{a\,d}{b\,c}$ is the quotient of $\frac{a}{b}\div\frac{c}{d}$; also, the numerator and denominator of this quotient are equal; that is,

$$a\,d=b\,c.$$

This last expression is perfectly general; because it includes incommensurables; and, which is of perhaps greater importance, it makes the most useful property of equal ratios, the one upon which their definition is founded. In words it may be expressed as follows:—

If four quantities are proportionals, the product of the first and fourth is always equal to that of the second and third. The first and fourth are usually called the extremes; and the second and third the means; and the definition may be also expressed, Four quantities are proportional when the product of the extremes is equal to that of the means. As this is a perfectly simple definition, it holds true conversely, so that we can either infer the equality of the ratios from that of the products, or the equality of the products from that of the ratios. This is the property of proportional quantities, which connects the general principles of proportion with the arithmetical operation of finding a fourth proportional to three given quantities; and this is the most useful operation in the whole practice of arithmetic. The common name for it is—

THE RULE OF THREE.

It gets this name because three quantities or terms are given as the data by means of which a fourth quantity may be found, which shall bear to one of those three the same proportion which a second of them bears to the third one. Thus, if a, b, c, are the three given quantities, and it is required to find a quantity x, in the meantime unknown, which shall bear the same proportion to c that b bears to a, we have the subject stated in its most general form ; and from what has been already stated, the four will be proportionals if

$$a\,x = b\,c;$$

and as these are equal, the quotient of $b\,c$, divided by a, must be equal to x; that is $x = \frac{b\,c}{a}$, or $a : b = c : \frac{b\,c}{a}$; which gives us the fourth term or quantity sought, in terms of the three which are given as data, expressed in words as follows :—

Problem.—To find a fourth proportional to three given quantities.

Rule.—Multiply the second and third, divide the product by the first, and the quotient is the quantity sought.

As the letters a, b, and c, which we have used in this simple investigation, are perfectly general, or may represent any quantity whatsoever, the rule itself is perfectly general, in all cases to which it can be applied. The next subject of inquiry therefore is, What are those cases? and when we have answered this question, we are enabled to turn the principle to every practical advantage. The conditions of the three given quantities, in order that there may be a proportion, are these :— First, that two of them shall be of the same kind with each other; secondly, that the one of these two shall stand in the same relation to the other, as the third given quantity stands to the quantity which is sought; and thirdly, that the third quantity shall be of the same kind with the quantity sought.

These are the general conditions which are absolutely necessary to bring the particular case within the problem; and in order to prepare them for arithmetical operation when so brought, it is necessary that the two given quantities, which are of the same kind, should also be in the same denomination; that is, that 1, numerically considered in each of them, should be exactly equal to 1 in the other; and also that the quantity of the same kind with that which is sought should be in the same denomination in which the latter is to be obtained. This last position will always hold without any preparation of the third quantity, because the denomination of it necessarily determines the denomination of the result.

Any quantities whatever which can be expressed by corresponding symbols, can be so managed as that the fourth proportional may be expressed in terms of those symbols; and any quantities which can be expressed arithmetically, may be so managed as that the fourth proportional may be obtained in the same denomination as the given one which is of the same kind with it; and if a different denomination of it is necessary, this can be procured by the common arithmetical methods of changing the denominations of quantities without changing their values.

There is still, however, another consideration necessary for the expert use of this operation; and that is the proper arrangement of the three given quantities. One principle in this arrangement consists in having the first and second quantities of the same kind with each other, and the third of the same kind with that which is sought.

This arrangement is easily made, because the kinds of the quantities is a matter which can hardly be misunderstood. But there is still some difficulty in knowing which of the two quantities that are of the same kind should occupy the first place in the arrangement. This however may be obviated in all cases by attending to the following

Rule.—Write down as the third term that quantity which is of the same kind and in the same circumstances as the quantity sought. Next consider whether, from the nature of the case, the quantity sought ought to be greater or less than this third quantity. This is the part of the subject on which a little judgment is required; but unless we are able to decide the point, we are neither in a condition for arriving at the quantity which we seek, nor for knowing whether it is the right one after we have obtained it. We shall suppose, therefore, that the case has been duly considered, and that the conclusion is, that the quantity sought must be greater than the third quantity. In this case the greater of the two given quantities, which are of the same kind with each other, must occupy the second place in the arrangement, and the less the first place. On the other hand, if the quantity sought ought to be less than the given quantity of the same kind, the less of the two given quantities which are of the same kind with each other, must occupy the second place in the arrangement, and the greater of them the first place.

If these simple directions are attended, to there never can be the smallest difficulty or mistake in the arranging, or, as it is called, the stating of three quantities, which are given for the purpose of finding a fourth proposition: nor is it difficult to say that such must be the case; for if the first term of the one pair of quantities is less than the second, the first of the second pair must also be less than the second of that pair; and also if the first of the one pair is the greater, so must the first of the other pair be the greater, otherwise in neither case the one pair would express a ratio of majority, and the other a ratio of minority, while the expressed condition of all such operations is, that the ratio of the first to the second shall be exactly the same as the ratio of the third to the fourth, otherwise the product of the second and third divided by the first, would not express the value of the fourth, or quantity sought.

A very simple case of a practical nature will serve to illustrate this:—Forty-eight men performed a piece of work in nine days; and it is desired to have another piece every way equal performed in twelve days, how many men must be employed for the purpose?

Here the quantity sought is a number of men, and therefore the given number of men 48, must be the third term. Next, will the required number of men be greater or less than 48? On looking at the numbers of days, we find that 12 days are to be allowed, whereas only 9 days were allowed in the former case; and therefore the number of men required must be less than 48, it being presumed that each man performs equal labour every day in the one case as in the other; for without this consideration being understood, the two numbers of men would not be in the same circumstances, and therefore not proportionable quantities. The statement, or arrangement of the given quantities, will therefore stand thus:—

Men.	Men.	Days.	Days.

$$12 : 9 = 48 : x,$$

x being merely a sign for the number of days which is sought.

After the terms are stated in this manner, it is desirable to examine and see whether the first term, and either the second or the third have any common divisors, because if they have, they may be divided by those divisors and the quotients used in their stead, upon the principle already explained, that like parts of quantities have exactly the same quotients as the quantities themselves. Looking in this way at the above quantities we find 12 the first term and 48 the third, both divisible by 12, and the quotients are 1 and 4. Hence we have—

$$1 : 9 = 4 : x;$$

therefore $x = 9 \times 4 = 36$, the number of days sought.

It very often happens that it is as useful to multiply the terms of a proportion as to divide them; because by this means they may be cleared of fractional parts; and the principle is, that the first term must be multiplied by any number which multiplies either the second or the third term; and if both the second and the third are multiplied either by the same number or by different numbers, the first must be multiplied by the same number for every time that either of the others is multiplied. When this is done, the fourth number, when obtained, will always be in the same denomination in which the third was originally stated.

There is one particular case of simple proportion—that is, of the comparison of two equal ratios with each other—which deserves a separate notice; namely, that case in which the second and third terms are equal to each other; or when this common valuer of the two may be considered as a *mean proportional* between the other two. This may be stated,

$$a : b = b : c;$$

and the equal products in this case are,

$$a\ c = b^2.$$

In this case if any two are given, the third can be found. Thus, $a = \frac{b^2}{c}$; $c = \frac{b^2}{a}$; and $b = a\ c^{\frac{1}{2}}$. The last expression may also be written $b = \sqrt{a\ c}$. The sign used in this last expression is only another form of $\frac{1}{2}$, and it will be more particularly explained in the section on the powers of quantities. The rules expressed in words are as follow:—To find the mean proportional, take the square root of the product of the extremes; and to find either extreme, divide the square of the mean by the other extreme.

COMPOUND RATIOS OR PROPORTIONS.

The ratios, and the proportions or comparisons of ratios which we have been hitherto considering, have been viewed only as perfectly simple; but there are other ratios which are compound, or made up of simple elementary ratios; and as the measure or simple expression of a ratio is the quotient of the first term when divided by the second, and consequently (which is in fact the very same thing, only differently expressed) the multiplier which applied to the second term produces the first, it follows that all ratios whatever are multipliers; and consequently that all compound ratios must be compounds of multipliers; that is, products arising from the multiplication of those numbers which express the ratios or multipliers.

In this case, however, there is a distinction to be made between a ratio which is a common number, and one which is an exponent, or expressive of the number of times that a quantity is to be used as a factor,—the difference between which will be found explained on looking back to page 39, and more fully in a future section on the management of exponents. In the meantime we have nothing to do with exponents, and therefore we may regard the measures of the ratios simply as numbers.

This being understood, we may state that a compound ratio is one which is produced by the multiplication of two or more ratios; and that this multiplication may be either performed by multiplying all the antecedents into one product, and all the consequents into another, which two products will then express the compound ratio in one antecedent and one consequent; or the measures of all the separate ratios may be found and multiplied together, and their product will be the measure of the compound ratio.

To illustrate this we may take a case in numbers, and we shall take one in which the ratios considered as multipliers are all integer numbers. Let the series be,—

3, 6, 30, 150.

In this series the ratio of 6 to 3, taken as a multiplier of 3, is 2; the ratio of 30 to 6, taken as a multiplier of 6, is 5; and the ratio of 150 to 30, is also 5; therefore the compound ratio of 150 to 3 is the product of $2 \times 5 \times 5 = 50$. Inversely the ratio of 3 to 150 is the product of the reciprocals of the measures of those three ratios, or of 1 divided by each of the three numbers; that is $\frac{1}{3} \times \frac{1}{5} \times \frac{1}{5} = \frac{1}{150}$, which last is the reciprocal of the compound ratio viewed in the other way.

This is a very simple case, but it explains the principle just as well as the longest series and the most complicated ratios that could be introduced; therefore we have this general principle: If there is any series of quantities of the same kind (in which case they all necessarily have some ratio to each other), the first has to the last a ratio compounded of the ratios of the first to the second, the second to the third, and so on through the whole of the series; and inversely, that the last has to the first a ratio compounded of the ratios of the second to the first, the third to the second, and so on to the end of the series; and farther, that the compound ratio is expressed by the continued product of all the simple ratios in the one case, and of all their reciprocals in the other.

If in a series of this kind the ratios of the terms taken two and two in their order be equal to each other, the ratio of the first to the third is the square of that of the first to the second, which is technically called *duplicate* ratio; and the ratio of the first to the fourth is the cube of that of the first to the second, and is called a triplicate ratio. Also, the inverse of a duplicate ratio is the ratio of the square-root, and is ratio is a*sub-duplicate* ratio; and the inverse of a triplicate called a ratio of the cube-root, which is called a *sub-triplicate* ratio.

The compound ratios which we have now explained are understood to be among a series of quantities of the same kind only; and thus they are continuous between one term and another throughout the whole series, however long; but there are other cases in which the continuity is broken between every pair of terms; and a series of this kind does not express a perfect compound ratio, unless the number of terms in it is even, or consists of a number of pairs of terms; but it is of no consequence what the number of those pairs may be.

The reason of the interruption in a series of this nature is the quantities being of different kinds; and there is no correct compound ratio unless each pair which stand nearest to each other in the series are of the same kind with each other. If they have this last property, they can be compounded into one single ratio, that is into two terms, having a ratio equal in value to the product of all the antecedent terms for a general antecedent, and all the consequent terms; and from a series of ratios of this kind we get the principles for performing that arithmetical operation which is called

THE COMPOUND RULE OF THREE.

This is also sometimes called *compound proportion;* but that term is too general, inasmuch as it includes also those series which have the ratios continued from term to term. In these last there is no comparison of ratios, and consequently no means of judging of the equality of ratios or of finding any term; and consequently no operation in the least analogous to the rule of three can be performed by means of these.

With those disjointed proportions, which when complete consist of as many separate ratios as there are pairs of terms, and have each pair of the same kind with each other, and are arranged as antecedents and consequents in the same order, the case is very different; for if the consequent of the last pair is wanting, it can be found by the very same operation, and on the very same principles as in the common rule of three, only there is a little preparation necessary in order to bring it to three terms, so that the common rule can be applied.

But the preparation necessary is exceedingly simple, and consists of nothing further than multiplying all the antecedents of the complete ratios for a common or compound antecedent, and all the consequents of the same for a compound consequent; after which these two, together with the antecedent of the incomplete ratio, become three terms, by means of which the fourth term can be just as easily found as if the first and second terms had been simple at the first, or only one complete ratio had been given.

In arranging or stating the terms of those simple ratios, whose continued products are to be the terms of the compound one, exactly the same principle must be observed, as in stating the first and second terms of a simple proportion; and as the particular problems or questions which fall under this branch of the practice of proportion are of a more complicated nature than those in which only one complete ratio is given, they require to be examined with more attention. It may not, therefore, be amiss briefly to mention the precautions which are necessary.

First, it must be ascertained whether there is a number of pairs of terms, in which those of each pair are of the same kind and in the same circumstances with each other; that in addition to these there is a single term of the same kind and in the same circumstances with that which is sought; and that besides these there is no other term given; for unless the question has these conditions, it does not involve a proportion capable of being reduced to a compound first and compound second term, of the same nature and composition with each other, and a single third term of the same kind and in the same circumstances with that which is sought.

Secondly—The two terms of each of the complete simple ratios must be expressed in the same denomination, so that their numerical values may have the same ratio as their real ones; for if this is not done, then, though the whole arrangement and process may be rationally correct, the compound terms, and consequently the result, will be arithmetically wrong; and as the quantity sought is an arithmetical value, it will be as erroneous as though there had been similar errors in the rationale of the process.

These two points require a good deal of consideration; but this consideration is not only absolutely necessary, for it is also exceedingly well bestowed; because when the compound rule of three is once thoroughly understood, it becomes a most efficient instrument in the simplification and abridgment of difficult calculations, not only in the arithmetic of common business, but in every case where calculation can be applied, or indeed where one general conclusion has to be drawn from a number of relations, even if those relations are the points of an argument, or the stepping-stones by which one finds one's way to the accomplishment of a difficult project of any kind.

It is of no consequence of what kind the terms of any or of all the simple ratios may be, provided they are of the same kind; and it is of no consequence in what denominations or terms they are expressed, so that both are expressed in the same. For we may take the following as a general definition of compound terms :—

The terms of a compound ratio are of the same kind and in the same circumstances with each other, if all the simple ones of which they are composed, taken one in each product, have this property.

This is nearly self-evident; because all the conditions are given in the terms of the simple ratios; and as each of the compound terms is the result of an equal number of factors and multiplications, there is nothing in the operation which can change the relations of the terms. Further, in the arithmetical part of the process, the terms which have to be multiplied with each other may be taken in any order, because changing the mere order of the terms does not affect the value of the general product. In almost every case, too, the operation may be greatly shortened, by dividing the antecedents and consequents by their common measures; and in doing this it is of no consequence whether the terms that are divided by the common measure be those of the same simple ratio or not; because the general result will not be affected by this; and that result is the only thing which is wanted.

Every operation of this kind could be performed by as many successive operations in the simple rule of three as there are complete ratios given, by making the result of each operation the third term of the succeeding one; but in the solution of a complicated question this method is intolerably tedious; and in no case can those abridgments be made which are so advantageous in the compound operation.

The method of stating the terms is just a repetition of that formerly given; but still it may not be amiss to mention the rule very briefly.

Write down as the third term, that quantity which is of the same kind and in the same circumstances with the one that is sought. Take other two terms of the same kind and in the same circumstances with each other; consider whether on account of them the quantity sought should be greater or less than the third term; and having decided this, place the greater or the less of them for the second term accordingly. State all the remaining pairs in the same manner, ranging the antecedents under each other in a first column, and the consequents in a second, until all the pairs are exhausted. If there is no other number or quantity given, the conditions are properly stated; and, when they are reduced to the same denomination in each pair, the whole is rightly prepared.

There is another mode of arrangement, which though exactly the same as this, is perhaps better, because it brings all the data into smaller compass. It is this:—draw a horizontal line, and write the term corresponding to that sought immediately above it towards your left hand. Then write all the second terms of the different ratios after this, with the sign × between every two: write all the first terms after each other under the same line, with the same sign between every two; and the whole expression is a general fraction equal in value to the quantity sought, while nothing remains to be done but to reduce this fraction to its simplest form, or to its value as an integer when it has such a value.

We shall illustrate this method of operation by a very simple instance:—A log of wood imported from a certain part of the world cost 100*l.*; and the following are the particulars of it—the length 24 feet, the breadth 5 feet, the thickness 3 feet 9 inches, the duty 5 per cent., and the freight and charges 25 per cent.: another log of wood has been imported from a different part of the world, of which the following are the particulars,—the length 30 feet, the breadth 4 feet, the thickness 3 feet, the duty 10 per cent, and the freight and charges $12\frac{1}{2}$ per cent. Also a solid foot of the second log is worth, in the country where purchased, 3 times as much as a solid foot of the first log: it is required to find the price of the second log.

The fractional statement of this is as follows:—

$$\frac{100 \times 30 \times 4 \times 3 \times 10 \times 12\frac{1}{2} \times 3}{24 \times 5 \times 3\frac{1}{4} \times 5 \times 25 \times 1} = \text{the price of the}$$

second log.

Draw another line, and see what can be shortened. The one 100 is a manageable number, and therefore it may be left alone, 30×4 above is equal to 24×5 below, so that these four numbers may be left out. Again, $10 \times 12\frac{1}{2} = 5 \times 25$, therefore they may be left out; and 3 above and $3\frac{1}{4}$ below, when both divided by 3, make 1 in the upper line, and $1\frac{1}{4}$ below: so that the remaining numbers are—

$$\frac{100 \times 3}{1\frac{1}{4}} = \frac{1200}{5} 240l. \text{ price of the second log.}$$

The particulars of the two logs, in this example, were taken much at random, and not with any very particular view to the shortening of the operation. In strict propriety, there should have been three operations in the calculation of this problem. The freight and charges are understood to be at so much per cent. *ad valorem* on the other costs of the two logs; and the duty at so much per cent. *ad valorem* on the total cost, freight, and charges included. Consequently, the other costs should have been first estimated, and then the freight and charges made the first and second terms, with 100 in each in addition to the rate per cent. The result of this should have again been made the third term, and 100 + the duty, in the two cases, the first and second terms; or these might have been compensated in the one operation, by making the terms each

100 + the per centage, which would have made the original fraction.

$$\frac{100 \times 30 \times 4 \times 3 \times 110 \times 112\frac{1}{2} \times 3}{24 \times 5 \times 3\frac{3}{4} \times 105 \times 125 \times 1} \text{ the price.}$$

This, shortened by throwing out all the equal products which are obvious, and dividing by all the common divisors which are equally so, becomes—

$$\frac{100 \times 22 \times 9}{1\frac{1}{4} \times 7 \times 10} = \text{price of the second log.}$$

Shortening this again, by dividing 100 by $1\frac{1}{4} \times 10$, which is $12\frac{1}{2}$, reduces the 100 to 8 in the upper numbers, and obliterates $1\frac{1}{4}$ and 10 in the under, we have—

$$\frac{8 \times 22 \times 9}{7}$$

which cannot be shortened; but, performing the multiplications in the upper term, we obtain

$$\frac{1584}{7};$$

and performing the division, we have

$226\frac{2}{7}l. = 226l.\ 5s.\ 8\frac{1}{2}d.$ nearly =

the true value of the second log.

We took the first of these methods, in order to show the modes of proceeding in cases where the data need no correction; and we have added the second in order to show how the correction of the data can be made; for if the three operations are performed in succession, the final result will be found to be exactly the same as the last arrived at by the general operation, namely $226\frac{2}{7}l.$

We shall give another example, in order to show the use of the compound rule of three as an instrument of investigation. Let it be required to find a formula for computing the interest

i of any sum of money p, for any number of days d, at any rate r of interest per cent, per annum, that is 100*l.* for 365 days.

From the case, it is evident that 100 and 365 must be the antecedents of the two ratios, p and d the consequents of the same, and r the last term; for r is interest, and therefore the only quantity in the same circumstances with that which is sought, though 100 and p are of the same kind with it. The arrangement of the terms will therefore be

$$100 : p = r : i \text{ the interest}$$
$$\text{and } 365 : d$$

If 365 is multiplied by 2 it becomes 730, which is a more simple number. Therefore multiply it by 2; and as r is always a small number, and will be simplified if it contains $\frac{1}{2}$ per cent., multiply it by 2, to compensate the other. Then the fractional expression for the operation will be

$$\frac{p\ d\ 2\ r}{73{,}000} = i, \text{ the interest.}$$

In words:—Multiply the principal sum by the days, and by double the rate per cent.; divide the product by 73,000; and the quotient is the interest. As it is best to perform all complicated operations in which money is the chief subject, by decimal arithmetic, from the ease with which the subdivisions of a pound sterling can be converted into decimals of a pound, or the reverse (as explained at page 69), the divisor, in this calculation, may be simplified by cutting off three additional places of decimals from the product, and dividing by 73.

Let us take an example. Required the interest of 468*l.* 17*s.* 6*d.* for 219 days at $3\frac{1}{2}$ per cent.—

$$\text{Here, } p = 468{\cdot}875$$
$$d = 219$$
$$2\,r = 7, \text{ therefore}$$

$$\frac{468{\cdot}875 \times 219 \times 7}{73{,}000} = 9l.\ 16s.\ 11d., \text{ the interest.}$$

RECIPROCAL PROPORTION.

If the product of any two quantities taken as factors be equal to the product of any other two different quantities also taken as factors, then from the definition of equal ratios, it follows that those factors are reciprocally proportional; which means that the first of the first pair has the same proportion to the first of the second pair, which the second of the second pair has to the second of the first pair. Thus let $a\ b$ be the product of a and b, of which a is the first factor and b the second; and let $c\ d$ be the product of other two factors, of which c is the first, and a the second: then

$$a : c = a : b.$$

For by hypothesis, that is, by the conditions assumed, the product of the extremes $a\ b$, is equal to the product of the means $c\ d$, and this is the essential property of proportionals. These proportionals are reciprocal, because the term of the first product stands first in the first ratio, and the term of the second stands first in the second ratio, or the terms of the one ratio are in reverse order to those of the other.

This is a simple principle, as depending entirely on a simple definition; and therefore it is convertible, or may be stated as a truth the other way; that is, if two factors and other two are reciprocally proportional, the product of the first pair is equal to the product of the second.

Simple as this principle appears, it is by no means an unimportant one, as will appear from a plain example. A man has a table 12 feet long and 5 feet broad, and he wishes to have another table 9 feet long which shall contain just as much surface as the first; what must be the breadth of the second table?

We have explained, in a former section that a surface is represented by the product of its length and breadth; and as the surfaces of those tables are to be equal, the products of their lengths and breadths must also be equal, consequently they are reciprocally proportional, that is, calling the unknown breadth x,

$$12 : 9 = x : 5$$

Hence, $x = \frac{12 : 5}{9} = \frac{60}{9} = 6\frac{2}{3}$, feet, the breadth required.

This case often occurs in questions solved by the common rule of three; but the common rule applies to it as well as to all others. It is useful, however, in many instances, and among others in

PROPORTIONS INVOLVING A CONSTANT QUANTITY.

We shall be best able to explain what is to be understood by this by taking an instance:—A mine constantly produces the same quantity of water in the same time. Upon one occasion it was neglected and allowed to get full, and it took 12 horse power 60 hours to empty it. On a second occasion it was again allowed to fill, and 10 horse power required 80 hours to empty it. Now it is full a third time; and it is desired to know what power will empty it in 48 hours, and also what power of an engine must be erected to keep it dry after it is emptied? It is to be understood that the horse power and the flow of water into the mine, are quite constant, that is, exactly the same in all the cases.

It is evident that the product of the number of horse power, and that of hours taken in each case, will give us the work done in each case, expressed in the same denomination in both, namely the work of 1 horse power in 1 hour. In the first instance, $12 \times 60 = 720$ hours' work; and in the second instance $10 \times 80 = 800$ hours' work. Comparing this with the former, we find that the smaller power working for the longer time, had 80 hours more work to perform; and this is our clue to the whole matter.

During 60 of the 80 hours there would come in just as much water as there did in the other 60, and the pitful is the same in both cases. It is clear, therefore, that the 80 hours' work, that is the water requiring 80 hours' work, is the quantity that came in during the additional 20 hours in the second instance; and if we divide it by 20 we shall obtain the quantity which comes in in one hour, in terms of the number of horse power necessary for removing it during the same: $80 \div 20 = 4$. Therefore, 4 horse power will remove in one hour the water which flows into the mine in one hour, consequently an engine of 4 horse power will be necessary for keeping the mine dry after it is once emptied, which is one of the answers to our questions.

Let us next inquire what power would be necessary for emptying it in 48 hours.

It is plain that whatever numbers of horse power is employed, and whatever length of time is required, there is always 4 horse power which does nothing towards emptying the mine, because it merely removes the water which flows in. This 4 is therefore the constant quantity which must be separated from the power in each of the two cases; and then the remaining powers must be reciprocally proportional to each other, because the product of each by the number of hours it requires is an expression for the same thing, namely the emptying of the mine, and so these products must be equal to each other. Taking 4 from 12, 8 remains; and taking 4 from 10, 6 remains; so that the mine was emptied by 8 horse power in 60 hours, and by 6 horse power in 80 hours; and the product in each of these cases is the same, namely 480.

Stating the terms, when thus cleared of the constant quantity as a reciprocal proportion, we have—

Power.		Power.		Hours.		Hours.
8	:	6	=	80	:	60.

Any one of these ratios will do for finding the power necessary for emptying the mine in 48 hours; and as it will take more power in 48 hours than in 60, the proportion will be

$48 : 60 = 8 : x$, the number required; and,

$$\frac{60 \times 8}{48} = 10$$

the power necessary to empty the mine in 48 hours; and if to this we add the constant quantity 4, we have 14 as the other answer to the question; namely, the power which in 48 hours will discharge the water already accumulated, and also that which comes in during the same time. The general answer then is: 14 horse power will be required to empty the mine, and a 4 horse power to keep it dry.

CHANGES OF PROPORTIONAL QUANTITIES.

In the changes of which we are about to speak, all the four proportionals are generally understood to be of the same kind; and the changes, if the definition of equal proportionals is properly understood, and the perfect correspondence of the two terms of a ratio to the divisor and dividend in division borne in mind, require little more than merely to be stated.

1.—Equi-multiples and like parts of the two terms of a ratio, produced by applying to them both the same multiplier or the same divisor, which may be any number whatever, have still the same ratio as the original terms. This follows immediately from the quotient not being altered by equal multiplication or equal division of the divisor and dividend.

Also, and for exactly the same reason, if like parts are added to the two terms, or subtracted from them, the sums in the one case and the remainders in the other will have the same ratio as the original terms.

This is the principle by the application of which we are enabled, for any practical purpose, to reduce the terms of ratios to their very simplest form; and it is, generally speaking, advisable to perform this simple operation on its terms, before the ratio is applied in any calculation in which those terms shall be blended with other quantities.

2.—Proportionals are proportionals, if taken *inversely*, that is, if the first and second and third and fourth are made to change places. This is evident, for the measure of the inverse ratio is the reciprocal of the direct one; and if quantities are equal themselves, their reciprocals are necessarily equal.

3.—Proportionals are proportionals if taken *alternately*, that is, the first has to the third the same ratio which the second has to the fourth. This is also evident, for the second is the first multiplied by the measure of the ratio, estimated as a multiplier, and the fourth is the third multiplied by the same; therefore, the second and fourth are equi-multiples of the first and third, and being so they necessarily have the same ratio. Inversely, the third has to the first the same ratio which the fourth has to the second; for the former are respectively like parts of the latter.

It does not follow in this case that though the second and fourth are considered as equi-multiples of the first and third, they should be greater than these; for the multiplier may be any number whatever, indefinitely greater than the number 1, or indefinitely less; when it is greater than 1, the multiple is greater than the multiplicand, and when less than 1 it is less. This follows from the very principle of multiplication.

4.—Proportionals are proportionals by *composition*; that is, the first or second has the same proportion to the sum of the first and second, that the third or fourth has to the sum of the third and fourth. This also follows from the principle of equi-multiples. The two terms are an equi-multiple of either term by the ratio of that term to the other × 1; and, therefore, the sums of the two terms are equi-multiples of the corresponding terms; that is, the first and third, or the second and fourth, and so they must be proportional.

5.—Proportionals are proportionals by *separation*; that is, the differences of the two terms of two equal ratios, taken in the same order, have the same ratio to the corresponding terms. This is still a matter of equi-multiples, for the difference of the two terms of a ratio is a multiple of either of them by the ratio of that term to the other — 1.

6.—Proportionals are proportional *ex æquali*; that is, when those which are equally distant from each other in one series of proportional quantities are proportional to those which are equally distinct from one another. This is still a matter depending upon the principle of equi-multiples; but it may perhaps require a little more explanation than the former ones.

Of this equality of ratios at equal distances in two series of quantities, there are two cases which very much resemble the direct and the inverse statements of common ratios; and these are usually cited in mathematical books by the words *ex æquo*, from equality, and *ex æquo inversely*, from equality of cross distance.

Ex æquo. If there are any number of quantities in one series, and as many in another series; and if these taken in the same order have the same proportion to each other two and two, it is assumed for proof that, when an equal number in each series is taken, the first of the last series will have the same ratio to the last of that series, as the first of the second series has to the last of the second series. Thus let

A, B, C, D, E, &c.

a, b, c, d, e, &c.

be two series of quantities in which $A : B = a : b$, $B : C = b : c$, $C : D = c : d$, $D : E = d : e$; then $A : E = a : e$.

The quantities in each series are continued proportionals, and their ratios taken two and two in order are the same; but we have already seen that in a continued proportion the ratio of the first term to the last is compounded of all the single ratios of the terms taken two and two; the terms in those two series are the same in number, and the ratios of every two terms nearest to each other are the same throughout. These individual ratios are the factors whose product forms the ratio of the first to the last; and those ratios are not only the same in value, but follow each other in the same order in both series; wherefore their products must be equal; that is,

$$A : E = a : e.$$

Ex æquo inversely. If there are any number of quantities in one series, and as many in another; and if these taken two and two in a cross order have the same ratios in the one series as in the other, it is assumed for proof that when an equal number in each series is taken, the first of the first series will have the same proportion to the last as the first of the second series has to the last. Thus, as before, let

A, B, C, D, &c.

a, b, c, d, &c.

be two series of quantities in which $A : B = b : c$, $B : C = a : b$, $B : C = c : d$, and $C : D = b : c$; then $A : D = a : d$.

This is a case of continued proportion as well as the former; and though the equal ratios in the two series do not follow in the same order, yet they alternate with each other, so that the whole in the one are still equal, each to each, to the whole in the other. But the ratio of the first quantity of each series to the last, is composed of the product of all the single ratios; and these being equal to each other, it follows that the products must be equal; because the order in which the factors are taken does not alter the value of the product. Consequently the product of those equal ratios arranged in cross order, are exactly the same as if they had been arranged in the same order in the one series as in the other.

The principles which we have now stated contain the whole elements of the doctrine of proportion; and any one who studies them with so much attention, as fully to understand them, can find no difficulty in managing any peculiarities which may arise in particular cases. Indeed, if the changes which can be made on the terms of a ratio without altering the ratio itself, and the condition which determines the equality of ratios, are once clearly understood, the whole doctrine of proportion may be said to be mastered. We shall therefore very briefly recapitulate these, because they are the points which it is essential for the student to bear in mind, in order to profit to the full extent by this most simple, most beautiful, and most useful portion of mathematical science.

1.—A ratio remains the same if both its terms are multiplied, or both divided by equal numbers, whatever may be the value and chararter of those numbers. It also remains the same if equal parts of the terms be added or taken away. In short, if whatever is done to the one term be also done to the other, and be done to both in the proportion of their original values, the relation of the terms, and consequently the ratio, remains unaltered amid all the changes, be they ever so many.

2. Ratios are equal, if the product of the first term of the first, and second term of the second, is equal to the product of

the second term of the first and first term of the second. And this equality holds whether the ratios are considered as original and simple, or as being compounded of any number of equal ratios. Compounding, in the arithmetical sense of the term, means multiplying the one by the other; and in all such cases, it is of no consequence in what order the factors are taken,—which last consideration is evident from the fact, that the 3 times 4 is exactly the same as 4 times 3.

If these two articles, and they are short and simple, are correctly borne in mind, the student will feel little difficulty in the management of proportion; and may and should turn to the Fifth Book of *Euclid's Elements*, as a most valuable subject of study, not for mathematical purposes only, but for laying a sure foundation for clearness and accuracy in general reasoning.

SECTION XIII.

POWERS AND ROOTS OF QUANTITIES.

A POWER of a quantity is the product which arises from the multiplication of that quantity by itself, the same quantity being both multiplier and multiplicand in the case of one multiplication, and multiplier in every successive multiplication, the product in the previous operation being multiplicand. A power is thus a compound quantity produced by two or any greater number of identical or equal factors; and different powers are distinguished by the number of times that the quantity of which they are the powers occurs as a factor in the composition of them. We have already partially explained some of the simpler properties of powers when treating of the scale of numbers, in the third section, and also in some other parts of the more elementary portion of this volume. We there mention that the arithmetic of powers gives occasion for a peculiar kind of numbers in arithmetic, and for a peculiar species of notation in algebra; and we shall here revert more particularly to the same subject.

We mentioned also that lines are arithmetically represented by simple or original numbers expressing their lengths in known measures; that surfaces are expressed in numbers by products of two factors, the one expressing the length and the other the breadth, expressed in the same manner, the number expressing the value in squares of the measure; and that solids, or those portions of space in which solids could be contained, are expressed in products of three factors, all in the same measure, the one being length, another breadth, and the third thickness. The number expressing the value of solidity in the last case consists of cubes, that is of solids bounded by six equal square surfaces, every side in all of which is the same measure as the denomination of lineal measure, in which the length, breadth, and thickness are expressed.

From these circumstances, a power arising from the multiplication of any quantity used twice as a factor, or once multiplied by itself, is called the square of the quantity. There is no objection to the use of this name square, whatever may be the kind of the quantity; because when the quantity is expressed by a number, the number might express the length of a line, and that line might be the side of a square; but it is only when the number actually means the length of a line that the product of it by itself is a square in reality. It is necessary to bear this in mind, in order that when we speak of the square of 3 or any other number, or of the square of *a*, or any other general quantity, we may not attach the same positive notion of squaring to it which we attach to the product of equal length and breadth, expressed in the same measure and multiplied together. Still there is a certain notion of squareness about the product of every number when multiplied by itself; for if we make as many dots in a straight line, as there are ones in the number, and repeat this line of dots at equal distances directly under the first one, the dots so made, if accurately placed, will form an exact square. Indeed, it is this very circumstance which makes the product of the length and breadth an accurate expression for the area or surface.

So also, when a quantity expressed by a number is multiplied twice by itself, or used three times as a factor, though the last product expresses a real cube only when the number represents a line, and when that line represents length in one instance, breadth in another, and thickness or depth in a third, yet there is some notion of cubism about the product of the same number used three times as a factor. For, just as the product of the two numbers when represented by dots makes a square, having the number of dots the same both up and down and crossways, and equal to the number of ones in the number in both, so the product of these numbers consists of as many of those squares of dots, as there are in each row of the square; and if we imagine these squares to be placed exactly over each other, and all at the same distance as the dots are both ways in the square, we shall have a cube of dots, containing as many dots as there are ones in the product of the number twice by itself. This cube is as comprehensible to the mind as if it were made of solid matter; and indeed it is this kind of cube which is referred to in our general reasoning on that solid,—not a cube of tangible matter, but the space which such a cube would fill if it really existed. In like manner the square which we speak of in our general reasonings concerning that form of surface, is not a real square portion of the surface of any substance which we could touch with our finger, or see with our eyes, but merely the extent and space to which such a surface could be applied.

Thus far geometry goes hand and hand with the general notion of quantity in algebra, and the particular adaptation to number in arithmetic. But when we get beyond the cube, geometry leaves us, inasmuch as extension, which is the proper subject of geometry, cannot be more than solid.

On the other hand, no geometrical quantity can be more simple than a line; because a point, which is the only consideration more simple, has no magnitude, and cannot be measured, or in itself made an element in any compound quantity produced by multiplication.

Algebra and arithmetic are not trammelled by the properties of extension; and therefore beyond the line, the surface, and the solid which belong to geometry in common with the rest, powers of quantities may be carried upward to any number of factors and multiplications, or downward by any number of repeated divisions by the factor, or multiplications by its reciprocal, which as already mentioned is only another name for division by the factor itself.

This factor—how often soever it may be repeated, it must be the same in every case, otherwise the product would not be a power—is called the ROOT of the other powers, because it is the value of this root which determines the value of all these powers; and it need hardly be stated that like powers of equal roots must be equal to one another, for they are the results of equal operations performed with equal quantities.

Any quantity or number may be regarded as a root, and the powers of it may be found upwards or downwards to any extent. Every number or quantity when not described as a power, is always in the same condition as a root; and in this sense of the term, a number is not necessarily considered as any other product than that of its own value by the number 1. Even in this sense, however, a number may be considered as a power; and thus considered, it is necessarily a first power, and its exponent, the general nature of which is explained at page 35, is ^{1}. The meaning of this is, that the number or root is considered as one factor only, and that there is no multiplication alluded to. Thus, if a is put for all quantities whatever, it is expressed as a power by a^1; but this differs in no respect from a without the exponent, further than by giving us a beginning for the powers of a. The other powers upward proceed in the natural order of the numbers, those numbers being written as exponents, and pointing out the number of times that the root is used as a factor, which is always 1 greater than the number of multiplications, the first multiplication requiring two factors, and the next lower power being always a factor in the one immediately above it.

From what has been already said, it is evident that the second and third powers are the only ones which can have names expressive of geometrical extensions; the second power being the square, and a^2 is read "a square," and the third power the cube, and a^3 is read "a cube." The fourth power is also sometimes called the biquadrate, because it is the square of the square, and perfectly intelligible in the management of quantities and numbers, though there is of course no geometrical magnitude which can answer to it. The other higher powers are named after the numbers of their exponents, and therefore require no further explanation. The following are some of the ascending powers of a, beginning with the first; and in as far as they are powers, and expressed by the exponents, they would be exactly the same whatever quantity or number was used instead of the letter *a*—

$$a^1,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,\ a^7,\ a^8,\ a^9,\ a^{10},\ a^{11}.$$

This series might be continued indefinitely and without limit; and when an indefinite exponent is introduced, the character for it is usually the letter ^{n}; thus a^n is the ^{n}th power of a, that is, any power indefinitely great, indefinitely small, or intermediate between these points.

Mere inspection of the above series of powers will show that every additional multiplication adds one to the exponent; and of course the addition of any number of times 1 to the exponent would express the same number of multiplications by the root. If we divide the product by the multiplier we always get back the multiplicand; and therefore any number of times 1 subtracted from the exp onent, expressesthat the power is to be as often in succession divided by the root, the quotient of each division being made the dividend in that immediately following. Thus a^{7-4}, means that the seventh power of a is to be divided four times in succession by a; but to divide by any number of divisors in succession produces exactly the same quotient as multiplying all the divisors and dividing by their product at once. Therefore a^{7-4}, is the same as dividing the seventh power of a by the fourth power of the same, which may be expressed by $\frac{a^7}{a^4}$, or by $a^{\frac{7}{4}}$.

If, instead of a^7, the first part of this expression had been simply a, then the remaining part of the expression would have been $a^{\frac{1}{4}}$. From this we can see that a negative exponent, and an exponent expressed as a denominator or divisor, have the same meaning, and are exactly equal to each other, when numerically alike; and thus a number of multiplications by the root, and a number of divisions by the root, or rather the root used a number of times as a multiplier, or a number of times as a divisor, are alike expressible by exponents; only there is this difference between them, that the exponent denoting multipliers has the sign + expressed or understood, and that expressing divisors has the sign — always expressed.

But from the perfect equality of the expressions $a^{\frac{7}{4}}$ and a^{7-4}, it follows that the negative exponent, or that which has — prefixed, is really the reciprocal of the same exponent when positive, or having the sign +; or that a^{-4} is exactly the same as $\frac{1}{a^4}$.

This at once shows us that + and —, as applied to exponents, have quite different meanings than when applied to quantities; and that there can in reality be no such operation as either the adding or the subtracting of exponents, in the sense in which those words are used with regard to common numbers and quantities. This leads to some very important conclusions, which we shall have to notice afterwards: in the meantime we may examine a series of those negative exponents.

Looking back at the positive exponents of a, we find that the left hand is 1, and that the others increase regularly toward the right, by the addition of 1 for every next succeeding term. If we look at them in the other order, we find that they diminish in order toward the left by the subtraction of one. Now we may readily continue the series towards the left hand; because a^1, where the former series stops in this direction, is any number or quantity whatever. If we divide that quantity by itself, the quotient must be exactly 1, whatever may be the value of a, because every quantity is contained just 1 time in itself, without anything deficient or over. If we divide another time by a, we have $\frac{1}{a}$; if we divide still another time by a, we obtain

$\frac{1}{a^2}$; and after that $\frac{1}{a^3}$, $\frac{1}{a^4}$, and so on as far as we choose to carry our series of divisions. These results, arranging them from right to left, are as follows—

$$\frac{1}{a}, \frac{1}{a^{10}}, \frac{1}{a^9}, \frac{1}{a^8}, \frac{1}{a^7}, \frac{1}{a^6}, \frac{1}{a^5}, \frac{1}{a^4}, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1}, 1.$$

Upon examining these from right to left, the first in the series is the number 1, whatever may be the value of a; and its exponent is ^{0}, that is, it has no exponent. This is exactly what might be expected; for the number 1 can neither be a multiplier nor a divisor. The next term in order is 1 divided by the square; that is, it is the reciprocal of the square, the third is the reciprocal of the cube, and the rest in order are the reciprocals of the corresponding powers in the former series.

But we have already seen that expressions in the form of the above are the same in import as if the exponent were made a fraction with 1 for its numerator, or written with the sign —. Hence we have,

$$a^{\frac{1}{11}}, a^{\frac{1}{10}}, a^{\frac{1}{9}}, a^{\frac{1}{8}}, a^{\frac{1}{7}}, a^{\frac{1}{6}}, a^{\frac{1}{5}}, a^{\frac{1}{4}}, a^{\frac{1}{3}}, a^{\frac{1}{2}}, a^0; \text{ and also}$$

$$a^{-11}, a^{-10}, a^{-9}, a^{-8}, a^{-7}, a^{-6}, a^{-5}, a^{-4}, a^{-3}, a^{-2}, a^0.$$

But though these answer to the same places in the general series, they do not express the same meanings. The first of the two series—namely, that with the fractional exponents—expresses roots of a considered as a number of which a root answering to the exponent is to be found; for the number 1 which occurs as the numerator of each shows that a is still to be taken as one factor, or rather as the power arising from the multiplication of the factor, which must be in all cases different from a. As for instance, $a^{\frac{1}{4}}$ means that some root or number r is to be found, which when used four times as a factor, or multiplied three times by itself, shall produce a result either exactly equal to a, or as nearly equal to it as possible.

In the other series the expressions are simply the descending powers of a, having that quantity for their root, and being in fact the reciprocals of the corresponding positive powers.

To take an example in numbers: let $a = 5$; and let us write the powers of a expressed by their exponents in one line; the powers of 5 expressed by their factors in a second line; and the same powers expressed by the products of their factors in a third line.

1st,—	a,	a^2,	a^3,	a^4,	a^5
2d.—	5,	5×5,	$5 \times 5 \times 5$,	$5 \times 5 \times 5 \times 5$,	$5 \times 5 \times 5 \times 5 \times 5$.
3d.—	5,	25,	125,	625,	3125.

The first of these three lines contains the general expressions for the first five positive powers of any simple quantity whatever, because the letter a may mean anything. The second line contains the first five powers of the particular number 5, expressed in terms of 5, the root connected by signs of multiplication which point out the operations to be performed. In each power the number of times that the factor or root 5 is repeated, corresponds exactly with the exponent of a in the first line; and the number of times that the sign of multiplication $\times$ occurs in each points out the number of multiplications. The third line has exactly the same value in numbers as the second, only the multiplications are performed, and each power appears as a separate number.

Let us next take the same illustration with negative exponents; and we shall begin them at the left in order that they may be the more easily compared with the former.

1st,—	a^0,	a^{-1},	a^{-2},	a^{-3},	a^{-4}.
2d.—	$\frac{5}{5}$,	$\frac{1}{5}$,	$\frac{1}{5 \times 5}$,	$\frac{1}{5 \times 5 \times 5}$,	$\frac{1}{5 \times 5 \times 5 \times 5}$.
3d.—	1.	·2,	·04,	·008	·0016.

The first of these lines contains the general expression of powers immediately following each other below the first power; and as the first of these is an expression for the number 1, it has the exponent ^{0}; the others are continual divisions of 1 as many times successively by a as there are ones in their respective exponents. The second line contains a particular instance of the same, in terms of the numerical root 5; and the third line contains the values of the expressions in the second stated in decimals,—that is, in terms of 10, the root of the common arithmetical scale.

Upon carefully examining this third line, it will be found that there is some information to be obtained from it, further than the mere illustration of powers with negative exponents. The root 5 made use of in the second line, is a factor of the number 10, and the number 2 is the other factor. Now if we examine the decimal numbers in the third line, we shall find that they are the powers of 2, divided by the corresponding powers of 10. We may leave out the first of the series because it is 1, and of no use as a power. The second ·2 is $\frac{2}{10}$; the third ·04 is $\frac{2\times 2}{10\times 10}$, for it is $\frac{4}{100}$; the third ·008 is $\frac{2\times 2\times 2}{10\times 10\times 10}$, for it is $\frac{8}{1000}$; and the fourth is $\frac{2\times 2\times 2\times 2}{10\times 10\times 10\times 10}$, for it is $\frac{16}{10,000}$.

From this it follows, that if any power of a number is divided by the same power of one of its factors or component parts, the quotient is always the same power of the other component part, and conversely. If like powers of any numbers of factors are multiplied together, their product will be the same power of the product of all the factors. Thus, if a and b are the factors of any quantity c, $a^2\ b^2 = c^2$, when the power of the composite number or quantity is found by multiplying the same powers of its factors, it is of no consequence in what order they are taken. Thus, $5^2 \times 2^2$, and $2^2 \times 5^2$ are both $= 10^2$; and it is the same in every other case however complicated.

The general explanation which we have here endeavoured to give of the elementary doctrine of powers, is sufficient to prepare an ordinary reader for more elaborate works, and so we shall now proceed to the practical matters of expressing the powers of numbers and quantities, and finding them by calculation in any case where that may be required; for this purpose we make a sub-section.

Involution.

Involution literally means "rolling up," in the same manner as multiplication means "manyfolding;" and the distinction of the names is worth attending to: because in multiplication there may be many factors all different from each other, the one of which is, as it were, folded down upon itself as often as it is expressed by the other; whereas in involution, or the finding of powers, the same quantity is, as it were, folded its own number of times upon itself at every multiplication.

The first point to be attended to on this subject, is the correct expressions of powers; and this is a matter of no inconsiderable importance; because, in complicated operations especially, it is desirable to get our expression for the whole operation to the end before we begin any of the parts of the operation, and thereby lose sight of some of the given quantities. Indeed, this is a matter of so very great importance, that it ought to be enforced upon every occasion where the enforcement of it can be introduced, even though it may appear not only inelegant, but apparently tedious. If we do not hammer it till it becomes hot, the spring becomes the more elastic the more that we hammer it; and it is even so with the human mind: if we wish to impart to it that vigorous elasticity which, once acquired, will send it bounding along the paths of knowledge in the strength of its own energy, so that we keep it cool, it is our duty to hammer away, in order that error or prejudice may not afterwards bend it from the line of the truth.

It is not in mathematical investigations alone that this practice of finding, the expression before proceeding to the operations, is one of the most useful rules that can be observed; for it applies equally to every project that can be accomplished, and every object which can be obtained; and we may add, that it is owing to the want of this preliminary seeing of the way and the means, more than to any thing else, that the failures and mishaps of mankind, in every department of life, are owing.

If the quantity of which any power is to be expressed is simple, or consists of a single number in arithmetic, or a single letter in algebra, the power is expressed at once by writing the proper exponent over the right hand of the number or quantity; and in this case, in numbers, the power itself can be found by common multiplication, the extent of which is pointed out by the exponent

In the case of compound quantities, a little more attention is necessary in expressing their powers. If, indeed, the whole of the compound quantity is to be considered as the root of which the power is sought, we have only to inclose it within parentheses, or draw a vinculum over it, and write the exponent as if it were a simple quantity. In algebra, an exponent occurring in a compound quantity is not understood to extend any further than the single letter over which it is placed. Thus $a+b^2$ is not the square of $a+b$, it is $a+$ the square of b. As little is a^2+b^2 the square of $a+b$, for it is the sum of their squares. The square of $a+b$ is $(a+b)^2$, or $\overline{a+b}^{2}$. If the compound quantity of which the power is to be expressed is a product, then the whole may be either enclosed in parentheses, or the exponent may be attached to each letter, or other factor. Thus, the square of a product of a and b is either $(a\,b)^2$, or $a^2\,b^2$. So also, if the compound quantity is a quotient expressed in terms of a dividend and divisor, the exponent may be either affixed to the whole, or to each term separately. Thus $(\frac{a}{b})^2$, or $\frac{a^2}{b^2}$, is the square of the quotient of a divided by b. If the terms of a compound quantity are connected by the sign $+$, or the sign $-$, they must always be enclosed in parentheses; because no multiplication or division extends to both sides of either of those signs, unless it is so expressed.

We have made these explanations as simple as possible; but they contain all the elements, and will apply to the most complicated quantity that can arise.

The next point for our consideration is the value of the power in terms of the root and the exponent. We do not, in the mean time, allude to the numerical value in any particular case, but merely to the general value, as to whether the power is to be greater or less than nothing, greater or less than 1, or greater or less than the root.

First, no power can be less than nothing, though any quantity not considered as a power may. $-a^2$ has meaning as part of many compound quantities; but standing alone it has no meaning, as it cannot be the product of any quantity by itself. The product of $+a$ by $+a$, and that of $-a$ by $-a$, are both exactly the same quantity—namely $+a^2$, as is explained at length at page 80; and $-a^2$, though it has meaning as the product of $+a$ by $-a$, thereby indicating that as many times a should be taken away as is expressed by the said a; yet a square can be obtained only by multiplying a quantity by itself; and instead of $+a$ and $-a$ being the same quantity, the difference of their values is $2a$. If, however, the exponent of a simple quantity is an odd number, and the sign of the root —, the sign of the power will also be —; so that the powers of negative roots, taken as single quantities, have the sign — in all the even ones, and the sign — in all the odd ones.

Secondly, if the root is greater than 1, the power must be also greater than 1 and greater than the root, and the power must increase as the exponent increases; but if the root is less than 1, the power must be less than 1 and also less than the root, and the power must diminish as the exponent increases. This, of course, applies only to positive exponents; for if the exponent is negative, the power must in all cases be less than one; for ^{0} is the exponent of the number 1 in the case of all roots whatever; and the negative powers are all less than this, and less exactly in proportion as the positive powers are greater, for they are the reciprocals of those powers.

That every power of a number greater than 1 must be greater than both 1 and the root, is very easily seen, because even in one multiplication it is taken or repeated as many times more than its original value as that value exceeds the number 1; and it signifies not how small this increase may be in any particular case, but it must be something. If we carry the matter onward another step, we have the increased number increased again by the same number of times itself as the increase; and if we pursue the powers in their order to any length, we shall still find that the argument applies, and that each succeeding power becomes greater than the one immediately before it.

Nor is this all, for let a root be ever so little more than 1, we could imagine with perfect correctness a power of it with so high an exponent to be taken, as that the value of this power would be greater than the value of any number that we could name. For instance, let ^{1} represent the smallest imaginable number, and ^{n} the greatest possible exponent that we can imagine, but not name; then $(1+{}^{1})^{n}$ is really greater than any number that could be arithmetically expressed, even though the line of figures extended both ways through space to the most distant stars.

It is equally apparent that no power of a quantity less than 1 can ever be so great as 1, or indeed so great as the root itself. For, if we take the square which is the result of multiplying the quantity or root by itself, we have a quantity less than 1 reckoned up, repeated, or taken less than one time itself, which is saying expressly, though in other words, that the square of a quantity less than 1 must be less than that quantity. If again we consider the second multiplication, which produces the cube or third power, we have this diminished quantity the square taken less than once itself; and if we continue to examine the successive multiplications which produce the powers in their order, we have the multiplicand less in each case than in the case before it, and the multiplier the same in them all, so that as the exponent increases, the value of the power diminishes.

When, however, we turn our attention to the descending exponents, or those which are less than 1, the index of the root, we find the state of things reversed. Whatever the fraction is, the value of its power which has ^{0} for its exponent, is 1, the same as in the case of all other roots; and as the next term is 1 divided by the fraction, which by hypothesis is less than 1, it follows that the quotient of 1 divided by it must be greater than 1; and as the divisors of 1 continue to decrease after this, the quotients must increase, for those quotients which are the values of the powers, are the reciprocals of the positive powers which have the same exponents.

A very simple case will serve to illustrate this; and the very simplest fraction we can employ for the purpose is the fraction $\frac{1}{2}$. The first five positive powers of this are—

$\frac{1}{2}$, $\frac{1}{2}\times\frac{1}{2}$, $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$, $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$, $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$.

Performing the multiplications we have,

$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$.

Or, expressing them in decimals, they are

·5, ·25, ·125 ·0625, ·03125.

Thus the positive powers of $\frac{1}{2}$ go on diminishing as the exponent increases, and each power is the same fraction of the one preceding it, namely $\frac{1}{2}$ of it; and in the case of any other fraction every positive power must be the same part of the one immediately preceding it, that the root or original fraction is of the number 1.

Let us next examine the powers which have negative exponents. The first six of them are as follows,—

$$\frac{\frac{1}{2}}{\frac{1}{2}},\ \frac{1}{\frac{1}{2}},\ \frac{1}{\frac{1}{4}},\ \frac{1}{\frac{1}{8}},\ \frac{1}{\frac{1}{16}},\ \frac{1}{\frac{1}{32}}.$$

If we suppose each of these to be multiplied by the denominator of the fraction in the under term, we shall thereby reduce all the lower terms to 1, while the upper ones will be the same as those denominators. Therefore in the case of the fraction $\frac{1}{2}$, the first negative term, that is the term after the one with the exponent °, is 2, the next is 4, the next to that 8; and so on, doubling the value for every 1 that the exponent is increased numerically, though diminished in value because of the negative sign. Those negative powers increase in a ratio corresponding to the reciprocal of the root; and it is evident the reciprocal of any other fraction must be the rate of increase in those terms which have negative exponents, for the same reason that the fraction itself is the ratio of decrease in those powers which have positive exponents.

Let us next consider the case of a fractional number, the value of which is greater than 1. Such a number may be regarded as made up of two parts—the one an integer, either the number 1, or any number greater than 1; and the other the fraction, which in all cases must be less than 1, or, if it is not so originally, the integral part of its value can be separated from the fractional part, and taken in with the other integers. Hence such a number, which may be called a mixed number, consists of two distinct parts, which, without reducing the integer to the terms of the fraction, cannot be added together so as to have their value expressed by one simple number. It may therefore be represented by $a+b$, in which a is the integral part, and b the fractional.

The expression of any power of it will be $(a+b)^m$; and that of its square, $(a+b)^2$. The composition of this square has been already alluded to at page 99; but we shall again repeat the operation:

$$(a+b)^2=(a+b)\times(a+b).$$

$$\begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \underline{\quad\; + ab + b^2} \\ a^2 + 2ab + b^2 = (a+b)^2. \end{array}$$

Though this composition of the square of $a+b$ is very plain, it may not be amiss to dwell a little longer upon it, for the purpose of fixing it in the minds of such readers as may not have previously studied the subject. Whatever may be the value of a, it is plain that the a multiplied by a must be a^2, so also the product of a and b must be both a times b, and b times a; and whatever is the value of b, the product of b multiplied by b must be b^2; then, as the product of a and b occurs twice, 2 can be made a co-efficient, which saves writing $a\ b$ twice over with the sign + between. The expression translated into words, and stated generally, is—

The square of the sum of two numbers is made up of, and consequently equal to, the sum of their squares and twice their product.

We shall now consider a as an integer, b as a fraction, and the two taken together as a mixed number. We can do this with perfect propriety; because a and b, being perfectly general expressions, represent all numbers whatsoever. Then according to the composition, as above explained, the square of the mixed number must consist of the square of the integral part, twice the product of the integral part by the fractional part, and the square of the fractional part. The square of the first is of course always an integer; twice the product of the two may sometimes be an integer, though generally speaking it will be a mixed number; but the square of the fraction, having its denominator the square of the denominator of the given fractional part, must not only be in every case less than the given fraction, but must be such that no fractional part, which may arise in twice the product of the integer and fraction, can be so applied to it as to make it an integer; wherefore, the square of no mixed number whatever can, under any circumstances, become a whole number. If this be true, as it evidently is, in the case of the square, much more must it be true in the case of any higher power, because in such a power the denominator of the power of the fraction is still more compounded. This is a very important conclusion, and so is the converse which follows immediately from it; namely, if any root of a whole number cannot be exactly expressed by a whole number, it cannot be exactly expressed by any mixed number whatever. We shall have, however, to consider this a little further, when we come to examine a few points in the doctrine and management of roots. In passing, we may remark, however, that the very simplest whole number which can have a root different from itself, namely, the number 2, belongs to this class. The number 1 is not the root of 2, for the square of 1 is only 1, and the square of 2 is twice 2, so that 2 cannot be its own root. Consequently 2 has no square root which can be expressed, either in a whole number or in a mixed number, with perfect accuracy; though there is no doubt that the square root of 2 is a real quantity, because the product of it by itself is the number 2.

The existence of roots not expressible, gives rise to a different kind of numbers from any that we have hitherto mentioned. They are called *surds*, or *irrational* numbers, that is, numbers of which the ratios cannot be exactly stated. At one time the management of these numbers was exceedingly troublesome; but since the introduction of Decimal Arithmetic, and various other improvements of modern times, they occasion very little trouble in practice, because their values can be approximated to any degree of accuracy that may be required.

The powers of all numbers and quantities, simple and compound, being merely the results of multiplications, performed in exactly the same way as other multiplications, it is not necessary to go into any explanations of the process. It may not be amiss, however, to show the composition of the cube of the sum of two numbers or quantities; because there are some elementary proceedings in which it is useful.

Let us then examine the cube of $a + b$, where we have,

$$(a + b)^3 = (a + b) \times (a + b) \times (a + b).$$

We have already obtained the square, and if we multiply this square by $a + b$, the product will give us the cube,

$$\begin{array}{rl} (a+b)^2 = & a^2 + 2ab + b^2 \\ \text{Multiply by} & a + b \\ \hline & a^3 + 2a^2b + ab^2 \\ & \qquad a^2b + 2ab^2 + b^3 \\ \hline & a^3 + 3a^2b \times 3ab^2 + b^3 = (a+b)^3. \end{array}$$

Upon examining this we find that it consists of four terms, the cube of the first quantity, three times the square of the first quantity multiplied by the second, three times the first quantity multiplied by the square of the second, and the cube of the second quantity.

This formula rarely occurs in ordinary business, compared with that of the composition of the square; but still it is desirable to bear in mind the composition of the cube, and indispensable to commit that of the square to memory.

It would be very easy to investigate any number of these formulæ, because for every higher one we have only to multiply another time by $a + b$, and translate the product into words. There is, however, a general method, by which any power of $a + b$ may be obtained without the trouble of performing the multiplications; and we shall very briefly state the leading principles of that method.

From the examples which we have had, it will be perceived that the square consists of three terms, separated by the sign +, and the cube consists of four terms, separated from each other by the same sign. If the operations were continued, it would be found that the number of terms in each power is 1 greater than the exponent of the power; and that the first term of the root only appears in the first term of the power with the same exponent as that of the power, while only the second term of the root appears in the last term of the power, also with the exponent of the power. Further, neither of those two terms has any numeral co-efficient, or number before it, and they are the only terms without such co-efficient. If we take the other terms from the beginning, the exponent of the first term of the root diminishes by one from term to term, or is 1 less in every term than in the term immediately to the left. On the other hand, the second term of the root does not appear at all in the first term of the power, but it appears in the second term without any exponent, and therefore with the exponent 1 understood; from this toward the right the exponent of the second term increases by 1 in every term toward the right, until in the last term, in which the first term of the root does not appear, it is the same as the exponent of the first term of the root in the first term of the power.

The exponents of the two terms of the root are thus, as it were, applied reversed upon each other, only the first extends a term further than the second at the beginning, and the second a term further than the first at the end. It will perhaps render this arrangement of the exponents more clear if we introduce them for a moderately high power, disencumbered of the letters, co-efficients, and signs, and merely separated from each other by commas; and it may be as well that we do this for a power which has the exponent an odd number, and consequently an even number of terms; and then for a power which has the exponent an even number, and consequently an odd number of terms. It is to be understood that in both, the first line expresses the exponents of the first term of the root a; and the second line the exponents of the second term of the root b.

First.—Exponents of a and b in the ninth power of $a + b$, or $(a + b)$.

9, 8, 7, 6, 5, 4, 3, 2, 1,
1, 2, 3, 4, 5, 6, 7, 8, 9.

Here it will be seen that the sum of the two exponents in every term is the same; and that it is the same as the exponents of the first and last terms, which are the same as that of the power.

Secondly.—Exponents of the tenth power of $a + b$, or $(a + b)^{10}$.

10, 9, 8, 7, 6, 5, 4, 3, 2, 1,
1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

In this case the sum of the exponents is equal in all the terms as it was in the former; but there is one peculiarity here, and that is that the exponents of both terms of the root are equal in the middle term of the power. This could not be in the former case, because there is no middle term there, the number of terms being even; but if we look back to it, and examine the two terms on each side of the middle, we shall find that the exponents of both are the same numbers, differing from each other only by the number 1; the first term of the root having the larger exponent in the first of them, and the second term of the root having the larger exponent in the second. From this illustration it would be an easy matter to write down the exponents of all the terms of any power of $a + b$; and as no more letters than a and b can possibly appear in any term, and as their product, each affected by its own exponent, is the form in which they always appear when both are in the same term, the only thing which is wanting to complete the expression of the power is the co-efficients; for as both terms of the root have the sign +, all the terms of the power must also be separated from each other by the sign +.

To determine the co-efficients is also a matter of no great difficulty; and we have a general means of getting at the sum of these for any power; for, as the value of the co-efficients is independent of the particular value of the letters in any case, the sum of the co-efficients of the same power, including those of the first and last terms, each counted as 1, is always the same. Well, let us take the case in which both a and b are each equal to the number 1. In this case $(a + b)^n$ must be equal to $(1 + 1)^n$, that is, to 2^n. This is obvious, because from both terms being 1, neither of them can in any way count in multiplication. The whole value must thus be found in the co-efficients; and as the powers are just the powers of 2, it follows that in any one power the sum of the co-efficients must be the same power of 2. In the first power or root it must be equal to 2, and it is so, for it is once each of the terms. In the second power or square, it must be 4, and it is so, 1 in the first term, 2 in the middle term, and 1 in the last term. In the cube it is 1 in the first term, 3 in the second, 3 in the third, and 1 in the fourth, which make 8. If in like manner we were to examine any of the others we should find the sum always the same power of 2.

The next point is to determine what the several individual co-efficients are; and as the investigation, which depends on the transposition of the letters, is rather long for our purpose, if fully worked out, and not satisfactory unless it is so, we shall state in brief the result of it. The co-efficient of the second term is always the same as the exponent of the power or of the first term; and the other co-efficients are found thus: multiply the co-efficient of a in the second term by its exponent in the same, divide by the number of terms which is 2, the quotient is the sufficient of the third term; for every succeeding one, multiply the co-efficient and exponent of a in the preceding term, divide by the number of terms from the beginning including that; and the quotient is the co-efficient of the term next in order. Continue this operation throughout all the terms; and if that of the last but one come out the same as that of the second, and the increase toward the middle is the same as the decrease after it, the whole of the co-efficients must be correct. It follows that they must be so; because, as with the exception of that of the second term they are all derived from each other, if an error is committed at any step of the calculation it must be continued to the end, and the last co-efficient but one will not come out the same as the second.

The doctrine of combinations, upon which the finding of these co-efficients in a great measure depends, is a very beautiful subject, and when entered upon in the proper manner, it is by no means difficult. It is not, however, a purely elementary doctrine; and besides its beauty, could not be felt without a greater breadth of explanation than we can devote to the whole subject of powers. Our omitting it is the less to be regretted as we require only one particular case; and, even if we had room to investigate that case, it would lose much of its beauty if taken out of its connexion.

From what we have stated, it is easy to find any power of a *binomial*, that is, of any root consisting of two terms, represented generally by a and b; and the number of terms, the order of the letters, the exponents, and the co-efficients will be the same whether the sign of the last letter b is $+$ or $-$. The only difference is in the signs, which are all $+$ in the case of $a + b$, and alternately $+$ and $-$ in the case of $a - b$. It is worthy of remark, as showing the perfect agreement of the most complicated algebraical operations, that if the powers are found by successive multiplying, all the odd powers of the negative quantity $-b$ have the sign $-$, and all the even powers have the sign $+$. This follows of necessity from what was explained when treating of multiplication, in Section V.; and it supplies the only other element necessary for obtaining the power of a binomial, with the second term of the root negative, without the trouble of going over all the multiplications.

In illustration of the whole matter, we shall combine the different steps of the finding of the sixth power of $a - b$:

First—there must be seven terms.

Secondly—the exponents are for a and b, as in the following lines—

a . . 6, 5, 4, 3, 2, 1,
b . . 1, 2, 3, 4, 5, 6.

Thirdly—inserting the letters we have,

$a^6,\ a^5 b,\ a^4 b^2,\ a^3 b^3,\ a^2 b^4,\ a b^5\ b^6.$

Next—to find the co-efficients: that of the second term is 6; that of the third term, $\frac{6 \times 5}{2} = 15$; that of the fourth, $\frac{15 \times 4}{3} = 20$; that of the fifth, $\frac{20 \times 3}{4} = 15$; the sixth, $\frac{15 \times 2}{5} = 6$; and if we try the co-efficient of the last term, we have $\frac{6 \times 1}{6} = 1$; so that the calculation is right. The co-efficients therefore stand as follows, including those of the first and last terms—

1, 6, 15, 20, 15, 6, 1.

Then for the signs: the exponent of b is an odd number in the second, fourth, and sixth terms, and an even number in all the others, so that we begin with the sign $+$, and the signs of the other terms are $-$ and $+$ alternately. The complete sixth power of $a - b$, therefore, is —

$$a^6 - 6 a^5 b + 15 a^4 b^2 - 20 a^3 b^3 + 15 a^2 b^4 - 6 a b^5 + b^6.$$

Any power of a binomial may be found in the same manner without the trouble of performing the multiplications; and formulæ might be investigated for finding powers of roots consisting of greater numbers of terms. These are, however, of comparatively little use for practical purposes; so that we shall now proceed to the more elementary principles of

EVOLUTION.

As a root is said to be *involved* in a power, so when the power is given, and we find the root of it, that root is said to be *evolved*, or "unrolled," and the process is called *Evolution*. It is also called *Extraction*, which means "drawing out."

As any number, or quantity expressed generally by Algebra, may be given as a root, of which any power is required, so any number or any algebraical quantity may be given as a power, of which any root is required; but here the parallel between the two cases ends. If the root and the exponent of the power are given, the power itself can always be found exactly by the common process of multiplication, and sometimes without that process, as we have seen in the case of the powers of a binomial. But a root can in no case be found by the simple application of any of the elementary operations of Arithmetic or of Algebra; and there are many cases in which it cannot be found exactly by any method.

A very little reflection will show the reason of this. The data for finding a root are nominally very nearly the same as those for finding a power, namely, the power and the exponent; but in reality there is a very wide difference between them. In finding the power, the only instrument with which we have to work is the root, which is a common number or quantity, and as such has a value; while the exponent points out the extent of the working—the number of times that the root is to be used as a factor—which also has numerical value.

But when a root is sought, there is only one real number or quantity given—the power; and the other datum, the exponent, merely points out the relation which the quantity sought must have to the quantity given. But we cannot work with a relation, either in Arithmetic or in Algebra, unless we have that relation expressed in terms of two quantities or numbers; and here there is only one, therefore we have no means of obtaining a general solution of the problem of finding a root which will apply to all cases.

This will perhaps appear more evident, if we put the problem in its most general and simple form. *Any* root of *any* simple quantity, is the most simple and the most general form in which we can put it; and the simplest mode of expression is—Required the nth root of a.

We can indicate this root, either by a fractional exponent, or by the radical sign; in the first way, the indication or expression of the root is $a^{\frac{1}{n}}$; and in the second it is $\sqrt[n]{a}$. But there is no operation of Arithmetic or of Algebra pointed out by any sign in either of those expressions,—no addition, no subtraction, no multiplication, no division, and no proportion; therefore, the expression points out nothing that we can do, though it indicates in brief terms what ought to be done.

Yet we cannot doubt that the quantity which we seek is a possible quantity, or that, in the case of every particular value of the given quantity a, any root of a must have one constant and definite value; for if it had not this, its square, for instance, namely, the given quantity a, would not have a definite value; for the square is the product of the root itself, and its value depends upon, and is determined by, the value of the root. There is, indeed, one case in which we can at once tell the value of all possible roots of a, and that is when $a = 1$; for then $a^{\frac{1}{n}}$, or $\sqrt[n]{a}$, is also $= 1$. But then this tells us nothing further than that 1 is a number by which we can neither multiply nor divide.

What, then, are we to do in this case? We know that there is a quantity of a fixed and definite value to be determined, and a standard by which to measure its value after it is determined; and yet we have no direct clue to the quantity itself. We shall best answer this question, in a general way, by putting another: What are mankind obliged to do in all cases where they know what they want, but do not know the way to get it? They *try*, and if one trial does not succeed, they try again and again; and it is this perseverance in trying which is the parent of all discovery.

Therefore, let us try; and as every trial must be made on a particular case, let us take a particular number, and endeavour to find a particular root of it. We shall take a small number and a small exponent for our first trial; and it would be useless to try a number of which we already know the root, because that would teach us nothing. We need not try the number 1, because all its roots are the same as itself; and we need not try the square root of 4, because we can see at once that it is 2. Let us therefore take the number 2, and endeavour to find the square root of it—that is, to resolve into one number unaffected by any sign, the expression $2^{\frac{1}{2}}$, or $\sqrt{2}$. [When a root is indicated by the radical sign, it is not necessary to write the exponent of the square root, though it is necessary to write that of every other; and the exponent over the radical sign is written simply as a number, and not as a fraction, as the fact of its meaning a root and not a power, is indicated by the sign.

When the sign is applied to a compound quantity, the vinculum part of the sign must extend over all the terms, or the terms must be included in parentheses, and the sign placed before.]

The square root of 2 must be greater than 1 and less than 2; so let us first try $1\frac{1}{2}$, and take the square of that. $1\frac{1}{2}$ is $\frac{3}{2}$, and $\frac{3}{2} \times \frac{3}{2} = \frac{9}{4} = 2\frac{1}{4}$, which is too much, so that the root is less than $1\frac{1}{2}$. Try $1\frac{4}{9} = \frac{13}{9}$, and $\frac{13}{9} \times \frac{13}{9} = \frac{169}{81}, = 2\frac{7}{81}$; which is still too much. Try $1\frac{4}{10} = \frac{14}{10}$, and $\frac{14}{10} \times \frac{14}{10} = \frac{196}{100}, = 1\frac{96}{100}$; which is too small, but not a great deal, as the square is only $\frac{4}{100}$, or $\frac{1}{25}$ of 1 from the truth. We have found, therefore, that the square root of 2 is between $1\frac{4}{9}$ and $1\frac{4}{10}$; that is, that it is less than the former and greater than the latter. This shows the method of proceeding; and as we do not in the meantime require the square root for any particular purpose, we shall not pursue the investigation further; but proceed at once to the method of finding

THE SQUARE ROOTS OF NUMBERS.

In this, as in all other matters connected with numbers, we must attend to the proportions of the scale of numbers, as explained in Section III.; and as it is too long to be repeated, perhaps the reader may find advantage by a reperusal of that section at this stage of the inquiry. We must mention, however, that the places of the number, whether integers or decimals, all represent powers of 10, the unit's place being the centre of the series, or the term which has the exponent ^{0}; and the other places have their exponents, the numbers of their distances from this place, + in the integers, and — in the decimals. The figures which, in any particular number, occupy those places are multipliers of the corresponding powers of 10. Thus, in the integer number 365·, the unit figure 5 is a multiple of 10^0 by 5; that is, of 1 by 5; the tens figure 6 is a multiple of 10^1 by 6, that is, of 10 by 6; and the hundreds figure 3 is a multiple of 10^2 by 3, that is of 100 by 3.

In like manner in the mixed number 38,75, 5 being in the hundredths place of decimals is a multiple of 10^{-2} by 5, that is, of $\frac{1}{100}$ by 5, or it is 5 hundredth parts; 7 in the tenths place of decimals is a multiple of 10^{-1} by 5, that is of $\frac{1}{10}$ by 5; and the integers follow the same law as in the former instance.

Thus, in every number, the radical values of the several places—that is, their values when occupied by the number 1—remain the same for the same place from the units, whether that place be upwards in the integer numbers, or downwards in the decimals; and as the extent of the scale is perfectly unlimited both ways, it becomes at once the most easy and the most efficient instrument of calculation that can well be imagined. The series of powers which forms the basis of it adapts it remarkably well for all matters connected with the arithmetic of powers and roots, only this portion of it does not apply to any prime numbers except the two factors of 10, 2 and 5. The multiplications which are necessary when the figures in the different places are greater than 1, are easily understood and managed, because none of them is greater than 9'. The power of 10 too, however high, or however low, is the simplest of all possible factors, being always 1 with as many 0s after it as brings the estimate down or up to the unit's place.

In order to apply this scale to the finding of roots, and of course to the square root among others, it is necessary to commit to memory the powers of some of the smaller numbers; and for the square root every one acquires a knowledge of them in learning a common multiplication table; and the square of 9, or 9 times 9, is the highest that is absolutely necessary. This being borne in mind, we may proceed at once to the application.

For this purpose let us take a particular number; for instance, the number 54756.

As the scale of numbers to which we must look for assistance in this matter consists of two parts—the geometrical series or powers of 10, which determine the number of figures; and the multiplications of the different powers which determine the individual figures—we must divide our inquiry into two parts:—First, how many figures shall be in the square root of our number? Secondly, what shall those figures be individually? When we have obtained satisfactory answers to these two questions, we shall be in possession of the square root of the number. Let us set the number apart in order that we may consider it.

54756.

First—How many figures shall be in the square root of this number? The largest square which we can have of any single figure, is the square of 9, or 81; and it consists of two figures. The smallest square that we can have of a number consisting of two figures, is the square of 10, or 100; and it consists of three figures. Therefore we must have the units and tens of the power to answer to the unit figure of the root. But every two figures of a number which are immediately beside each other, stand in the same relation to each other as if the right hand one were units and the left hand one tens; and therefore we must have the hundreds and thousands of the power for the tens of the root. In like manner for every other figure of the root we must have two figures of the power; excepting in the case of the left hand one, and there we can have only one, if the number of figures in the power is odd. We might have arrived at the same conclusion in another way.

The figures in a number vary from each other in their places by a ratio of 10, the square of 10 is 100, and the exponent of 100 is ^{2}, while that of 10 is ^{1}. Therefore, for every figure in the root, with the exception of the left hand one, there must be two figures in the square. It is easy to see why this should not necessarily apply to the left hand figure; for it depends upon the individual value of that figure itself, whether its square shall consist of one figure or of two. 1, 2, and 3, have their squares consisting of only one figure, but every integer number greater than 3 has its square consisting of at least two figures.

It will be readily seen that this application of the exponent of the root to the scale of numbers in the power, is not confined to the square root, but applies equally to all roots whatever; and therefore we may state generally,—

To find how many figures shall be in any root of any integer number, begin at the right hand of it and divide it into periods consisting of as many figures each as the exponent of the root, and there will be as many figures in the root as there are periods. If there is a portion of a period on the left hand of the entire ones, that must be considered as a period as well as the rest, and counted accordingly.

There is rather a curious inference to be drawn from this; namely, that we cannot, by any single and direct arithmetical operation, find a root answering to a fractional exponent which has both its terms greater than the number 1; because we cannot divide the figures of a number into fractional periods. Let us return to our number and divide it into periods. It is as follows:—

5′47′56.

Therefore, the square root of 54756 must consist of three figures; and had it been any other number or any other power, the number of figures in the root would have been determined in a similar manner. Thus, there would be two figures in the cube root of the same number, because when divided in threes from the right it makes but two periods; and there would be but one figure in the fifth root of it; for it contains only five figures, which answer to but one figure of a fifth root.

Our next inquiry is, the particular values of the figures; and as we determined the number of figures by beginning at the right of the number, we must determine the figures themselves by beginning at the left.

The first period of the power, or number, consists of the figure 5 only; and this 5 must contain the square of the first figure of the root. 5 is not a square, but the largest square that we can subtract from it is 4, the root of which is 2, so that we have 2 as the first figure of the root, and 1 of the first period, and also the two other periods remaining; that is, we have the number—

1′47′56.

There are still three periods in this remainder, so that if it had been the number originally given, it would have had three figures in its square root. But we must bear in mind that though the places of the figures in the root are powers of 10, answering to powers of 100 in the square, yet that 1 of each power of 10 is multiplied by one of the nine figures, and the three products are added to make up the entire root. We may leave out the last period, and consider the second one only; and as the root of the first two periods is the sum of two figures, which, independently of their individual values, stand to each other in the relation of units and tens, we may consider them as a binomial, $a + b$, of which the square,

$a^2 + 2\,a\,b + b^2$, is in the present example $= 547$.

Of this we have already obtained the value of a, which is 2 as a single figure, but 20 when another is put after it; and it will be found that we have subtracted 20^2, or 400, from the two periods as they stood in the original number.

$$\begin{array}{lr} \text{From} & a^2 + 2\,a\,b + b^2 = 547 \\ \text{Subtract} & a^2 \qquad\qquad\qquad = 400 \\ \text{Remains} & 2\,a\,b + b^2 = 147. \end{array}$$

If we examine this algebraical remainder, we find that both terms of it can be divided by b, and that the quotient is $2\,a+b$; and conversely, if we had divided it by $2\,a + b$, the quotient would have been b, which is the representative of the second letter of our root.

Let us next see in what relation to the scale of numbers the two terms of $2\ a + b$ stand. a is one figure, b is another to the right of it, and in the meantime our consideration does not need to reach any further. Therefore, a is tens and b units; or if we substitute our numerical value for $2\,a$, we obtain $40 + b$ for our divisor; but b must occupy the same place which 0 occupies in the 20, so we may understand this and leave the units' place of our divisor blank. We shall then have—

4) 147 (.

As the divisor 4 is tens, we must get it out of the dividend, leaving out the last figure; that is, we must find how often it can be got in 14, and we soon discover that it is three times; so that the numerical value of b is 3; and if we write this 3 as units, after the 4 tens in our divisor, it will be complete; and the rest of this step of the process will be like common division.

43) 147 (3 = b, the second figure.
129
16 remainder.

We have now got the first and second figures; $a + b$; and we have subtracted the square of their sum $= 400 + 129 =$ 529, from the figures of our number without reference to the last period; that is,

From	547
There are taken	529
Remains	16.

Or remains of the whole number,

16′56.

Here again it might appear that the root of this remaining part would consist of two figures, and that the first of these two figures would be 4; but it is the sum of 23 with another third figure as units which is the entire root of which we are in quest; and the 24 which we have already obtained, and which, as there is to be another figure after it, is 230, must enter into the divisor, and be multiplied by 2 as well as by the third figure which we have still to determine. This 230 may be considered as 24 tens, and this will stand in the same relation to the figure which we have to determine as tens stand to units in any other number. Therefore, 24 may now be called a, and the figure of which we are in quest b, and with this understanding we may again refer to the algebraical operation. The whole will now be—

$$\begin{array}{lrl} \text{From} & a^2 + 2\,a\,b + b^2 & = 54756 \\ \text{Subtract} & a^2 \qquad\qquad & = 529 \\ \hline & 2\,a\,b + b^2 & = \;\; 1856 \end{array}$$

Hence the divisor, in order to find b, is $2\,a + b$, or $2 \times 23 = 46 + b$, and a place to be left blank until b is determined: so that the divisor and dividend will stand thus:—

46) 1856 (.

The 46 must be got out of the dividend exclusive of the units, because there is another figure to add to the divisor, and we at once see that the quotient cannot be more than 4. Hence the complete divisor and the dividend stand thus :—

$$\begin{array}{r} 464\ )\ 1856\ (4 = b\text{, 3d figure.} \\ \underline{1856} \\ 0 \text{ remains.} \end{array}$$

Therefore, the three figures of the root are respectively 2, 3, and 4; and the root itself of course is 234. If we take this and multiply it by itself, we shall find that the product coincides exactly with 54756; and in the same manner the square root of any integer number consisting of more than two figures might be found; but if the power consisted of two figures, or of one figure, we should have no means of finding the root but by trials, as in that case it would be only a single figure. But as 9 is the limit of roots of this kind, and as 1 requires no consideration, the squares of numbers from 2 to 8 are all that we require to try for by a single operation.

The principle which is here applied was alluded to in the early part of the volume, and we reverted to it again in the preceding section, when treating of squares. We may now mention it again with regard to any two parts of a number which stand to each other in the relation of units and tens, bearing in mind at the same time that if any one figure, whatever its place in the scale may be, is considered as units, all the figures to the left of it, taken as a whole expression, may be considered as tens. With this understanding, the arithmetical application of the square of a binomial may be stated as follows :—

The square of the sum of the units and tens is composed of the square of the tens, twice the product of the tens and units, and the square of the units. Of course, after the square of the tens is taken away, the first part of the divisor for finding the units is double the tens, and in comparing this with the dividend in order to find the quotient, which is the units of the root, the unit's figure of the dividend must be left out; and when the unit's figure has been determined, it must be placed on the right of the divisor as well as in the quotient. When the divisor thus completed is multiplied by the quotient figure, and the product subtracted from the dividend, this step of the operation is performed.

Collecting all these elements together, we have this general formula for the extraction of the square root of any integer number consisting of more than two figures:—

First,—To find the number of figures in the root: Begin at the right hand of the given number, and divide it into periods of two figures each; and the number of those periods, counting the single figure on the left, if there should happen to be one, shows the number of figures that must be in the root.

Secondly,—To find the figures. For the first figure, ascertain, by trial, the greatest square which can be subtracted from the left hand period of the root; the root of this square is the first figure of the root, and the square itself must be subtracted from the period, and the second period placed after the remainder, which thus increased will become the dividend from which the second figure of the root is to be obtained.

For the divisor: double the figure already found, and, leaving the unit's place blank, ascertain how often it can be had in the dividend exclusive of the units. This being done, place the number of times as the second figure of the root, and also as the unit's figure of the divisor.

Multiply the divisor thus completed by the figure of the root, and subtract the product from the dividend, annexing the third period to the remainder for a third dividend. Repeat this operation till all the periods are exhausted, always making the divisor double the root already found, and with the unit's place blank for the additional figure.

If, at any step of that operation, it is found that the first part of the dividend—namely, double the part of the root which has been found—cannot be got once in the dividend exclusive of the unit's figure; then 0 must be written in the root, and also in the divisor, and another period annexed to the dividend.

Should there be a remainder after a figure of the root has been obtained for each period, it shows (if the number of which the root is taken is wholly an integer number) that the root cannot be expressed exactly in terms of the common scale of numbers. But it may be approximated in decimals to any degree of accuracy; and the method of doing this is to annex to each successive remainder two 0s, as an additional period of figures. When the operation has been carried to a great length there is a large portion of the divisor which is not affected by the figures subsequently found; and then, a few additional figures may be obtained in the manner of common division, by leaving out an additional figure on the right of the divisor in the obtaining of each.

Decimal numbers are arranged according to the same scale as integer numbers, and consequently their roots are found in the same manner. But when the numbers contain no integers, the exponents of all the places in it are reciprocal or negative; and therefore the square roots are one degree higher than the powers; and in consequence of this we must count our periods from the decimal point to the right, or the opposite way to which we count them in integers. Also, if any number of periods at the commencement of the decimal, or next the point, consists of 0s, we must place one 0 in the beginning of the root for every such period. The roots of such decimals are always, of course, of greater value than the decimals themselves.

We have entered fully into the analysis of this operation, because it is one with which an intimate acquaintance is highly necessary; but having done so, we shall not need to adduce any further example.

When it is necessary to find the roots of fractions—which is, however, seldom done in ordinary practice—the method is to extract the root of each term; and this follows as a matter of course from the principle of the multiplication of fractions. If numbers, of which we require to find the roots, are expressed in factors, the root of each factor may be taken separately; and the product of those individual roots will be the root of the whole, just as the product of the factors is the whole power. It is not, however, desirable to do this, unless the factors are complete powers of which the roots can be exactly taken; because by these separate operations we should get an irrational number for every factor not a power, whereas by previously multiplying the factors altogether we should have only one irrational root, and thus could carry on our approximation with less labour and greater accuracy.

There is one case, however, in which it is often desirable to separate the factors of a number, in order to find the root of the rational one, and indicate that of the irrational by means of the radical signs. This is desirable only in the case of two factors; and the finding of the square root of 18 will serve as an illustration: 18 has several sets of factors; but two out of the number are 9 and 2, and 9 is a square: therefore we simplify $\sqrt{18}$, by expressing it $3\sqrt{2}$; which expression means 3 multiplied by the square root of 2.

There are various methods of expressing the approximations of the roots of numbers which are not complete powers, by continued fractions, and other infinite series; but they are not strictly of an elementary nature, and our limits will not admit of those explanations which would be necessary for rendering them pleasant or useful, or indeed intelligible; we shall, therefore, proceed to point out the means of finding

THE CUBE ROOTS OF NUMBERS.

It would be very easy, from mere inspection of the corresponding power of $a + b$, and the comparison of it with two numbers standing to each other in the relation of tens and units, to investigate a separate formula for finding every possible root. But this, though a good exercise for those who wish to discipline themselves in the practice of calculation, is not necessary in any case where the exponent is a composite number; because, if roots answering to the component parts of that number are taken in succession, the root of the one power being considered as the power in the next, the same object may be accomplished with more ease. Thus the 4th root is got by extracting the square root twice; the 6th root by extracting the square and cube roots in succession; the 8th root by extracting the square root three times; the 9th root by extracting the cube root twice; and so on in the case of all composite exponents. Where the exponents are prime numbers, there is no means of extracting the root by common arithmetic, except by a formula for the particular case. These large exponents rarely occur in practice, however; and when they do occur, there is a far simpler method of dealing with them, which we can better explain in the next section; so that we shall confine the few additional observations which we have to make upon this branch of the subject to the extraction of the cube root only.

In this, as in the extraction of the square root, there are two distinct considerations: first, what shall be the number of figures in the cube root of any given number; and secondly, what these figures shall be.

We have already mentioned how the number of figures in any root of any integer number is to be found, namely, by beginning at the right hand, or unit's place, and dividing into periods, each consisting of as many figures as the number expressed by the exponent of the root, with the exception of the period on the left, which must of course consist of whatever number of figures is left, after the other periods are marked off; and as the exponent of the cube root is 3, the periods of it must consist of three figures. If the number is wholly a decimal, the only difference is, that we must begin the periods from the left hand, and not from the right, as in integers; and if the number consists both of integers and of decimals, both must be divided into periods from the decimal point, the integers toward the left hand and the decimals toward the right. Also, in this and in every other root, if the right hand period of the decimal does not contain the right number of figures, the deficiency must be supplied by 0s when that period is annexed to the remainder.

In the second part of the operation, the finding of the figures of the root, the root of the left hand period, with which we begin, whether in integers or in decimals, must be found by trial, as in the case of the square root, because there is no device or contrivance which can assist us in finding any root when it consists of a single figure only. There are only eight cubes to be committed to memory for this purpose, namely, the cubes from 2 to 9 inclusive. The cube of 2 is 8, so that if the first period is less than 8, its cube root must be 1. The cube of 9 is 729, which of course is the largest which we have occasion to apply to a first period, and the others intermediate between these are easily learned, or easily found if not learned.

When the greatest possible cube that can be subtracted from the first period is determined, the root of that cube is to be taken as the first figure of the root, the cube itself subtracted from the period, and the second period annexed to the remainder; the remainder thus enlarged being the dividend from which the second figure of the root is to be found.

The next step of the operation is to find the divisor which we are to apply to the dividend for the purpose of ascertaining the second figure of the root; and here, as in the case of the square root, we must have recourse to the composition of the power of a binomial, $a + b$, though this is a little more complicated than in the case of the square root.

The complete cube of $a + b$, as formerly found, is,—

$$a^3 + 3a^2b + 3ab^2 + b^3.$$

But we are already understood to have found and subtracted the numerical value of a^3; and therefore the quantity which we have still to subtract, in order to get from the number the entire cube of $a + b$, is,

$$3a^2b + 3ab^2 + b^3.$$

This expression is the product of the divisor by b, the second figure of the root, whose value we are seeking; and therefore, if we divide this product by b—that is, if we lower the exponent of b by 1 in each term—the remainder, or rather the quotient, will become our divisor for finding b. There are three terms in this product, with the first power of b in the first term, the second power in the second term, and the third power in the third term; so that if we diminish the exponent of b by 1 in each, there will remain the divisor, which is,

$$3a^2 + 3ab + b^2.$$

a is tens, and b units, in relative place with regard to each other. Let us see, therefore, what are the places of the three terms. The first has a^2 in it, and the square of tens is hundreds;

therefore, the co-efficient 3 of the first term is 300, and the whole term means 300 times the square of a, that is, of the figure already found. The second term contains the first power of a, which is tens, affecting both the co-efficient 3 and the other factor b; therefore the second term is 30 times the product of the figure already found, and the figure which we are finding. At this stage of the operation, therefore, the value of b must be determined by trial; and it may take two or three trials of a beginner before the right one is discovered, because we are not yet in possession of the whole divisor. When the value of b is found, it must be multiplied by a and then by 30, and made the second line of the divisor, properly arranged under 300 times the square of a, the first figure, which is the first line of the divisor. The third term of the divisor is the square of b; and in respect of the other lines of the divisor, the right hand figure of it is units, because b is units, considered in relation to a as tens, and therefore the right hand figure of any power of it must be units, because the powers of quantities must be of the same kind with the quantities themselves.

Thus the complete divisor for finding the second figure of a cube root consists of three lines. First, 300 times the square of the figure already found; secondly, 30 times the product of the figure already found and the figure which we are finding; and thirdly, the square of the figure which we are finding. These three lines are to be added together, and their product by the figure which we are finding, subtracted from the dividend; and the figure is the second one of the root; while the remainder with the next period annexed makes a new dividend, out of which to find the third figure.

At the third and every subsequent step of the operation, all the part of the root found, which always stands in the relation of tens to the next figure as units, is to be used as a, and the figure we are finding as b; and with this understanding every step of the operation, however long it may be, is an application of the same formula.

If the root does not terminate in integers, so as to leave no remainder, it will not terminate in decimals; but it may be carried to any approximate degree of accuracy that may be required, by annexing three 0s to every remainder, and continuing the operation as before.

If the number is wholly decimal, we must begin with three figures immediately after the decimal point; and as often as three 0s occur between the decimal point and the significant figures, one 0 must be placed after the decimal point and before the other figures in the root.

The cube roots of fractions may be obtained by taking those of the numerator and denominator; and those of numbers expressed by factors, may be taken in the cube roots of the factors, and multiplied together for the general root. But as cubes occur much less frequently in the natural order of the numbers than squares, it is seldom that any advantage can be gained by this method. It is sometimes, however, of advantage to take the cube root of one factor of a number, and express that of the other by the radical sign. Thus, the cube root of 54, that is $\sqrt[3]{54}$, may be expressed by $3\sqrt[3]{2}$; for 54 is the product of 27 and 2, the first of which is the cube of 3.

We shall now give one short example, as illustrative of the application to numbers:—

Let it be required to find the cube root of 12812904?

Beginning at the right, and dividing into periods of three figures, because the exponent is ^{3}, we have

12′812′904.

The nearest cube to 12, the first period, is 8, the root of which is 2; therefore 2 is the first figure of the root, and its cube subtracted from the given number leaves

4′812′904.

We have now subtracted the cube of a, and there remains 4′812, out of which to find the second figure. a and b stand to each other in the relation of a tens and b units, therefore the divisor, as found by the general operation, is

$$300\,a^2 + 30\,a\,b + b^2;$$

and we must apply this to our number, and determine b by trials. $300\,a^2$ is the largest part of our divisor, so that we may compare it with the dividend,

$$a = 2 \text{ and } 300\,a^2 = 1200 \mid 4'812.$$

We would get 1200 four times in 4800; but the remainder 12 would not be equal to the cube of 4, and we want another term which is $30\,a$, or 60 times the number we are seeking, therefore we must try 3 for the value of b, our second figure; and as it is probable that 3 may answer, we may complete our operation.

	$300\,a^2$	=	1200	4′812
	$30\,a\,b$	=	180	
	b^2	=	9	
Sum =	$300\,a^2 + 30\,a\,b + b^2$	=	1389	
Multiply by	b	=	3	
Product =	$300\,a^2 b + 30\,a\,b^2 + b^3$	=	. . .	4167
Remains				645

We have now found 2 and 3 for the first and second figures of the root, 2 considered as tens, and 3 as units; we have subtracted the cube of their sum, which is 23, from the first and second periods of our number, and there remains 645. If we next annex the next period of our number to this, we shall have the following number for a new dividend,

645′904.

a is now $= 23$, but with this exception our divisor is exactly the same as before; therefore, using the numbers in place of the letters, we have $300 \times 23^2 = 158700 =$ the trial divisor. Comparing this with the dividend, we have

158700 | 645′904.

Comparing 15 with 64, which are the corresponding figures of the divisor and dividend, there being an equal number of figures to the right hand of them in each, we find it can be got 4 times, so that we may try 4 as our next figure; and the complete operation, using these numeral values in place of letters, will be

$300 \times 23^2 =$	158700	·645′904
$30 \times 23 \times 4 =$	2760	
$4^2 =$	16	
	161476	
Multiply by	4	
		= 645904
Remains		0

Consequently, 234 is the cube root of the given number 12812904; and the cube root of any number might be found in a similar manner; only when there are decimals in the number, this must be pointed from the left, while the integers are pointed from the right.

Any one who first examines the investigation of the formula with sufficient care, then goes over the steps of this operation, and bears in mind that the divisor is

$$300\,a^2 + 30\,a\,b + b^2,$$

and also that a means the figure or figures of the root already found, and b the figure which one is finding, can have no difficulty in extracting the cube root of any number whatever. Formulæ for all other roots may be found by taking the corresponding power of $a+b$, omitting the first term, diminishing the exponent of b by unity, or 1, in each remaining term, and bearing in mind that there are always as many 0s on the right of the numeral co-efficient in each term as there are terms to the right of that one in the formula.

SECTION XV.

ARITHMETIC OF EXPONENTS—LOGARITHMS.

In the case of every root or power, that is, of every quantity considered either as a root or as a power—and any quantity whatever, when spoken of and expressed generally, may be considered in this manner—there are three distinct subjects, any one of which may be the object of our inquiry. First, there is the root; and if the root is given, the power is found by multiplication, as already explained. Secondly, if the power is given, the root may be found by the methods explained in the latter part of last section. But thirdly, there is the exponent, and it also may become that which is sought; or, a and b being any quantities or positive numbers whatsoever, and ^{n} any exponent whatsoever, an answer may be demanded to the question, what power a is of b, or, in other words, to find such a value of ^{n} as that

$$b^n = a.$$

The exponent n may be any number whatever, integral or fractional. If it is a fraction, with 1 for a numerator, the power required will be a descending one, or a root; and if it be a fraction with both its terms greater than 1, a power must first be found answering to the numerator, and then there must be taken a root of that power answering to the denominator.

But when we begin to consider how we shall find such a value of n as will make $b^n = a$, we find that n is not an expression to which common arithmetic will apply, even though it should be expressed by a number. It is not a quantity, neither can it be made the representative of a quantity; because it means a number of factors, or rather a number of times that the same factor is to be used in multiplication; and this is a kind of expression upon which we cannot perform any of the common operations of arithmetic. If we add to it, we merely say that there shall be as many more multiplications as the number which we add; if we subtract from it, we merely say that there shall be as many multiplications fewer than there were before, as is expressed by the number which we subtract; in other words, that we are to *unmultiply* the number, that is, to undo a certain number of multiplications which enter into the composition of the number; and there is no means of undoing a multiplication but dividing the product by the multiplier.

Thus, in $b^{n+m} = a$, we are to multiply b^n as often by b as the letter m expresses, which of course may be any number of times, integral or fractional. Also, in $b^{n-m} = a$, we are to divide b^n by b as often as the exponent m expresses, which also may be any number, integral or fractional. Hence, though we cannot either multiply or divide exponents themselves, yet we can, by means of them, deal in a very summary manner with those numbers of which they are the exponents. For it follows immediately from what has just been stated, that,—

The addition of exponents is exactly the same as the multiplication of those numbers of which they are the exponents; and the subtraction of exponents is exactly the same as the division of those numbers of which they are the exponents. In other words, the sum of any number of exponents answers to the continued product of those numbers of which they are the exponents, and the difference of two exponents is the quotient arising from dividing the number answering to the exponent which we subtract from it.

Consequently, if we can get all numbers expressed by exponents, we shall get rid of the laborious operations of multiplying and dividing in our arithmetic, by substituting instead of them the comparatively simple ones of adding and subtracting.

Nor is this the only, nor even the greatest, simplification of our proceedings that we should obtain, if we could express all numbers by means of exponents; for it is evident that doubling an exponent is an expression for squaring the number of which it is the exponent; that multiplying an exponent by 3 answers to the cube of the corresponding quantity, and that multiplying an exponent by any number m is the same as raising the quantity to that power. But the multiplier m of it may be any number whatever, integral or fractional; and if it is a fraction with 1 for numerator, it is evident that the result is the same as dividing by the denominator of that fraction; and that if an exponent is multiplied by such a fraction, the product will indicate a root answering to the denominator. Thus $a^{n m}$ is the mth power of a^n; and $a^{\frac{n}{m}}$ is the mth root of a^n. These expressions are perfectly simple and general, and being such, they apply equally to all numbers, and all exponents whatsoever;

and hence again we have the following results from the use of exponents.

To find any power of an exponent, multiply the given exponent by that of the power required; and to find any root of any exponent, divide the given exponent by that of the root which is required.

It is for these reasons that, as has been hinted at in Section III., while endeavouring to explain the arithmetical notation and scale of numbers, exponents are called LOGARITHMS, or "the voices of numbers;" and there is something in the word *logos*, of which the first part of this name is composed, which is always worthy of our attention when we meet with the word alone or in a compound. *Logos* is not the mere sound—the noise in the ear made by an uttered word; for the Greek expression for that is *phone*, which means "a noise," and is equally applicable to all noises, whether there is any sense in them or not. *Logos*, on the other hand, is the information which the thing alluded to is capable of giving in answer to our inquiry or our observation; and therefore, as there may be a sound or *phone* where there is no *logos*, so there may be a *logos* without any *phone*, in which the information may be communicated to the eye or any other of the senses, or to the mind, without any instrumentality of the senses, which is in fact the case in our general perception of the meaning of ratios. *Logos* is nearly synonymous with ratio, and with relation; for as it is impossible for the mind to judge of anything without a standard of judgment, mentioned or implied, there is reference to such a standard in every case where truth or correct information of any kind is acquired.

Hence, in the plainest language of mathematics, Logarithms are the ratios of numbers; the expressions for Logarithms, whether given in the common figures of arithmetic, or in any other way, are the exponents of those ratios; and the operations which we are enabled to shorten, or perform in cases where common arithmetic is unequal to them, constitute the Arithmetic of Exponents.

In order to make this arithmetic available to the case of all numbers, and therefore available in any practical instance that may occur, we must first obtain the ratios of all numbers expressed in the common characters of our arithmetic. But in every ratio there must be two quantities of the same kind; and consequently we must have some one number; and our logarithms must be the expressions of the ratios of all other numbers to this one. The number 1 will not do for this purpose, because every number in its common arithmetical sense expresses the only ratio that it can have to the number 1; and as 1 can neither be a multiplier nor a divisor, those relations of the common numbers in arithmetic to the number 1, are simple arithmetical relations. But the relations of ratios which we require for logarithms, are geometrical ones; and when we seek for the logarithms, we seek for an answer to this question. How many times must a given number *m* be multiplied by itself, in order that the last product may be equal to any one number of common arithmetic? Separate answers to this question, in the case of every number, would furnish us with a series of logarithms, extending both ways without end, in like manner as the scale of numbers extends. But as the continuation of the arithmetical scale is obtained by the simplest of all possible operations—namely, by adding 1 to get the next higher number, and subtracting 1 to get the next lower; and as the logarithms, except those of very particular numbers, cannot be obtained without very laborious operations, we are obliged to rest satisfied with a limited number of logarithms,—in the common tables, with the logarithms of all numbers not exceeding 10,000.

The number m, which is one term in every ratio expressed by a logarithm, and which is of course the same in them all, otherwise we could not compare the ratios, is usually termed the *modulus*. This modulus is quite independent, both of the number compared with it, and of the logarithm which expresses the result of this comparison; and therefore it may be any number whatever. The number 10, being the foundation of the geometrical series, or order of the different places in our scale of numbers, is the best one to use as the modulus of our logarithms, and it is adopted accordingly. Logarithms formed by any modulus, employed as one term in all ratios, are called a system of logarithms; and the logarithms in common use are a system, having 10 for the modulus.

The modulus, or radical number, being once fixed upon, all the exact powers of it can have their logarithms expressed without any calculation, for they are whole numbers; always equal to the number of times that the modulus, or radical number occurs as a factor, or 1 greater than the number of multiplications. Thus, for instance, 10 being the modulus, we at once have, beginning at the left hand,—

Numbers	1,	10,	100,	1000,	10,000,	and so on,
Logarithms	0,	1,	2,	3,	4,	and so on;

and these give us the only simple integer numbers which can occur as logarithms, between the number 1 and the number 10,000 inclusive.

We can, however, carry the same series downwards below 1 in the powers which have negative exponents; and, also beginning at the left, they will stand as follows,—

Numbers	·1,	·01,	·001,	·0001,	and so on,
Logarithms	$\overline{\ }1$,	$\overline{\ }2$,	$\overline{\ }3$,	$\overline{\ }4$,	and so on.

The powers, marked numbers, in the last of these expressions, are the reciprocals of the same in the first, that is, they are quotients arising from dividing 1 by the corresponding powers of 10; and as the subtraction of logarithms is the same as the division of numbers, the logarithms in the second expression are also the reciprocals of those in the first. The logarithms have, therefore, the very same relation to each other in the two expressions as the numbers have; consequently, if the numbers, which are found by common multiplication, and can easily be verified, are correct, the logarithms must also be correct.

This fundamental part of the system of logarithms, as compared with the powers of 10, both ascending and descending, of which they are the exponents, is worthy of some consideration, as seen in its simple form, and without the other logarithms and numbers which have to be interpolated in order to complete the system of logarithms and the series of the natural numbers. In the first place, the number of figures in each of the integer numbers or ascending powers, is in every case 1 more than the units in the exponent; and each of the numbers is the smallest possible that can be expressed by the same number of figures; hence we have this general conclusion: the integral part of the logarithm of any number consisting of integers, or integers and decimals, is always 1 less than the number of integer places. Also, on looking at the decimal numbers or descending powers, we find that the integral parts of their logarithms always express the distance of the first significant figure from the decimal point, or that there are as many 0s before the significant figures of the decimal as the number of 1s in the integral part of the logarithm wanting 1.

Secondly, there must be as many logarithms between each of the two numbers in the line marked logarithms, as there are natural numbers between the natural numbers which answer to these. Thus, between 0 and 1, in the logarithms of integers, there must be eight logarithms, answering to the natural numbers from 2 to 9 inclusive; and these must be wholly fractions, and are best expressed by decimals. Between 1 and 2 in the logarithms, there must be eighty-nine logarithms, answering to the natural numbers from 11 to 99 inclusive. Between 2 and 3 there must be eight hundred and ninety-nine logarithms, answering to the natural numbers from 101 to 999 inclusive. Between 3 and 4 there must be eight thousand nine hundred and ninety-nine logarithms, answering to the natural numbers from 1001 to 9999 inclusive. If we take the expression for the descending powers, the number of logarithms between every two must be the same as between the corresponding two of these 1s.

Thirdly, the whole of those logarithms which require to be interpolated between the integral ones, whether those integers have the sign + expressed or understood, or the sign — always expressed, are necessarily fractional numbers; but none of them can be accurately expressed by any fraction, whether that consists of a numerator and denominator both expressed, or of a decimal, in which the denominator is understood to be 1 with as many 0s annexed to it as there are figures in the decimal, including in this number any 0s which may be between the significant figures and the decimal point.

This will appear evident when it is considered that, as already explained, no fraction and no mixed number consisting of an integer with a fraction annexed, can have any assignable power of it in a whole number. But in the case of a system of logarithms, of which the modulus or root is 10, and the natural numbers a series of powers, all the powers which are not integral, or consist of 1 with 0s after it, must be fractions; and as both root and power are whole numbers, and the exponent a fractional value, it is necessarily a fractional value which cannot be accurately expressed by any fraction whatever, just as it is impossible to express by any fraction whatever an integral power of an integer number which is not a complete power. Therefore, every logarithm which we can have, from 2 to 10,000 inclusive—and there are of course 9999 of them—must be irrational numbers, with the exception of 1, 2, 3, and 4, which answer to the first four powers of 10.

We can, therefore, express those irrational exponents or logarithms, only by approximation; and the simplest way in which this can be done is by using decimals, which in tables of even moderate accuracy are extended to six places, which express the value of the logarithm to the nearest millionth part of the number 1 in the integral portion of a logarithm; and as the differences of these logarithms form a series, though a diminishing one, a little more accuracy can be obtained by means of the differences.

Upon looking back at the succession of integer logarithms, it will at once be seen that the differences must diminish as the logarithms themselves increase; and that, as the numbers answering to the logarithms increase by multiplications by 10, while the logarithms increase only by additions of 1, the differences of the logarithms of high numbers, equally distant from each other in the scale, must be very much smaller than those of low numbers at the same distance in the scale. Between 0 and 1 there are only eight logarithms, and the natural number answering to logarithm 0 is 1, and that answering to logarithm 1 is 10; so that between logarithm 0 and logarithm 1 there are only nine differences. But between logarithm 1 and logarithm 2, there are eighty-nine differences, and yet their sum is only 1, as in the former case; so also between 2 and 3 there are eight hundred and ninety-nine differences, the sum of which is also only 1; and between 3 and 4 there are eight thousand nine hundred and ninety-nine differences, the sum of which is still no more than 1.

Now it requires no reasoning to show, that if the same general difference 1 has to be divided into 8 parts in one case, into 89 parts in a second, into 899 in a third, into 8999 in a fourth, and so on, the individual part must be very much less in each of those cases than in the case before it, even supposing that the parts in each case were all equal among themselves.

But there is nothing peculiar about those numbers in the arithmetical scale of which the logarithms are integers, further than that they happen to be integral powers of 10, the modulus or radical number of the system. For, the series of natural numbers goes on regularly by the addition of 1 to every lower number, in order to make up every one next following it; and the logarithms, though they express the exponents of those numbers or their ratios to 10, must still be true to the numbers, otherwise they could not be used as the representatives of those numbers; therefore, the difference between one logarithm and the one coming after it, goes on gradually diminishing, or becoming less and less, as we get to the logarithms of numbers higher in the scale.

This, though apparently a simple matter, is one which is very important to a clear understanding of the nature of logarithms, and a due appreciation of the advantages resulting from the use of them; and therefore we shall very briefly consider it in another light. Logarithms, and consequently the differences of logarithms, bear a very close resemblance to the quotients of numbers when divided by each other, and the differences of those quotients; and a very little consideration will suffice to show, that if the arithmetical difference between two numbers remains the same, the quotient arising from the division of the one of those numbers by the other must always be the less, the greater the numbers are, it being understood that the smaller number is the divisor. Thus, if the numbers are 1 and

2, of which the difference is 1, the quotient upon dividing 2 by 1 is the number 2; but if the numbers are 100 and 101, the difference of which is also 1, the quotient upon dividing the greater by the less is only 1·01, which is 99 hundreth parts of 1 less than the former. So, if the numbers are 1000 and 1001, the quotient arising from dividing the greater by the less is 1·001, which is only 1 and one thousandth part. If we went on increasing the equal parts of the two numbers by multiplication by 10 for a great number of times, and still had the difference only 1, we should at last arrive at a quotient differing from 1 by a fraction less than any fraction which we could name; and this gives us some idea of the rate at which the differences of logarithms diminish, as the natural numbers to which they answer increase.

But this difference between one logarithm and another, is not a quantity which can be added or subtracted upon the principles of common arithmetic, inasmuch as addition and subtraction are not operations which can be performed with logarithms, so as to have any meaning with reference to the natural numbers answering to the logarithms; for, as we have already shown, the addition of logarithms answers to the multiplication of the natural numbers for which they stand, and the subtraction of logarithms answers to the division of the natural numbers.

Logarithms do not represent quantities of any kind, except in that secondary way in which they are the representatives of common numbers; and therefore, when we are performing any operation by means of logarithms as the representatives of numbers, we must always find the represented number which answers to the representing logarithm, before we can perform either addition or subtraction. So also we cannot, by the addition of any difference which we can find by the principles of arithmetic to the logarithm of any number, get the logarithm of the next number above it in the common arithmetical series; neither can we, by the subtraction of any difference from the logarithm of any number, get the logarithm of the number next below it in the scale. If the numbers are composite, or the products of factors, we can get their logarithms by adding the logarithms of the factors; but if they are prime numbers, not the products of factors, we cannot get their logarithms in any other way than by a separate operation for each, and this operation is a very laborious one.

Thus, if the thing demanded of us were to find the logarithm of 2, we could get no assistance whatever from the integer logarithms 0 and 1, between which it lies, farther than that it must be greater than the one of them and less than the other; but how much greater or how much less, we have no direct means of finding out. What we want to know is, how often the number 10 must be multiplied by itself in order to make the number 2; and as 2 is a smaller number than 10, the exponent or logarithm of 2, of which we are in quest, must be fractional.

If we call the logarithm of 2 by the letter n, then our expression will be

$$10^n = 2;$$

and that which is required, is to find a fractional value of n, which will satisfy the conditions that are demanded of us. If we assume any fraction, and raise 10 to the power of the numerator, and 2 to the power of the denominator, we shall make a trial; and according as the power of 10 is greater than the power of 2, or less than it, we shall know whether the fraction we have assumed for the exponent is greater or less than the truth; for we have already stated that the truth is an irrational fraction, and cannot be expressed by any numerator and denominator whatsoever.

Let us first try $n = \frac{1}{2}$; then $10^1 = 10$, and $2^2 = 4$, which is

6 less than 10; so that the fraction $\frac{1}{2}$ is too great. Next try $n = \frac{2}{7}$, and we shall have $10^2 = 100$, and $2^7 = 128$, which is 28 more than 100; so that the exponent n is less than $\frac{1}{2}$, but greater than $\frac{2}{7}$. $\frac{3}{10}$ is a fraction intermediate in value between $\frac{1}{2}$ and $\frac{2}{7}$, let us again try it. $10^3 = 1000$, and $2^{10} = 1024$, which is still greater than 1000; and consequently the fraction $\frac{3}{10}$ is still too small; but as the denominator of that fraction is 10, if we are to express the exponent or logarithm decimally, the first figure of it will be three, and this is one approximation, though a rude one, to the logarithm of 2.

The next step of your operation would be to find the second figure, and here it might be as well to begin with 0 in the enumerator; and the figure which brought the power of 2 nearest to the power of 10, would be the second figure. Not more than nine trials would be required for this purpose; but the finding of the power of 2 would be a very laborious matter. Let us take $n = \frac{31}{100}$, and we have,

$$10^{31} = 10000000000000000000000000000000;$$
$$2^{100} = 336077764624128673834978317376;$$

which is less than the power of 10; so that $\frac{31}{100}$ is too large a fraction, and the second figure of the logarithm expressed decimally must be 0, or the first and second figures ·30, which is another and nearer approximation.

We might try $n = \frac{301}{1000}$, in order to find the third figure; and if the power of 2 were greater than the power of 10, the fraction would be too small; and we might try again with $n = \frac{302}{1000}$, and if the power of 2 were in this case less than the power of 10, then 2 would be too large for our third figure, and the first three figures would be ·301, which would be another step in the approximation, still nearer the truth than the former one. Multiplying 2 by 2 repeatedly, till it occurred a thousand times as a factor, would however be intolerably tedious; and if we were to try a step beyond this, it would be ten times more so. There are shorter methods than this of getting the logarithm of a prime number, but this is the only one which proceeds upon the common principles of arithmetic. The others are not quite elementary, or, at all events, they depend upon principles which we have not hitherto investigated; and therefore the introduction of them here would be foreign to our purpose, which is, to state nothing but what we can trace from first principles.

In an elementary point of view, this is a matter of but little consequence, because nobody requires actually to find a logarithm by an original investigation; for they are already found, and inserted in the common tables; and it is quite enough for every practical purpose, that we understand well the nature of a logarithm, can know how to find it, if that were necessary, by even the most tedious process, and are also acquainted with the structure of the tables, so as to be able to find in them either the logarithm answering to a number, or the number answering to a logarithm.

The decimal part of the logarithm of any figure, being derived wholly from that figure itself, and depending altogether upon its individual value, without any regard to the place which it occupies in the scale of numbers, is of course the very same whatever is the place of the figure, and whether it is integers or decimals. Thus, ·301030 is always the decimal part of the logarithm of 2, whether that 2 mean units, thousands, hundreth parts, or any other number whatever which is the result of multiplying or dividing 2 successively by 10; and it is the same with the decimal part of the logarithms of all the other figures. That such must be the case is quite evident; for multiplying by 10 is only adding 1 to the integral part of the logarithm, and dividing by 10 is only subtracting 1 from the integral part; and how often soever 1 is added to the integral part or subtracted from it, neither operation affects in the least the value of the decimal part.

Hence, when the integral part has the sign —, that sign does not affect the decimal part, which is always to be considered as a positive number; and the —, when thus placed before the integral part of a logarithm, expresses not an arithmetical but a logarithmic subtraction, which, as has already been shown, is the same thing as an arithmetical division; and as the subtraction 1 is the expression for a division by 10, — before the integral part of a logarithm merely points out that the figure answering to the logarithm, and considered as units, is to be divided as often in succession by 10 as there are 1s in that part of the logarithm which is affected by—.

Thus, the integral and the decimal parts of a logarithm relate, in different ways, to the number answering for the logarithm; and as, on this account, it is convenient to give a particular name to the integral part, it is called the *index*, or characteristic of the logarithm; and the decimal part only is called the logarithm, by way of distinction. When there is no sign, or +, before the index, it is said to be positive, and when there is — before it, it is said to be negative; but + and — here do not mean addition and subtraction; for we have already shown that these are operations which cannot be performed on logarithms. + means as many multiplications, and — as many divisions, by 10, of the corresponding number, as there are 1s in the index which it affects.

The index and the logarithm answer, therefore, to the two parts of which a number expressed arithmetically is made up. The index points out the distance of the left hand figure of the number from the units figure, and when it is + that figure is higher than the units, and when it is — it is lower. Thus, +3·

is always the index of thousands, and — 3· is always the index of thousandth parts; and it is the same with every other exponent. It is always 1 less than the number of figures in an integer number, and 1 more than the number of 0s at the beginning of a decimal one.

But though the index thus determines the place of the left hand figure of the number, and consequently also the places of all the other figures in their order, it gives us no information whatever with regard to the individual values of the figures; for whether the figure is 1, or 9, or any of the intermediate ones, the index of its logarithm is the same; so that the individual values of the figures answering to a logarithm depend wholly upon the decimal part, without any regard whatever to the index.

One who had not reflected on the subject, might be apt to suppose, that, as the logarithm of a single figure, disregarding the index, is the same whatever place in the scale that figure occupies, the logarithm of a number consisting of several figures could be found in some way from the logarithm of those figures taken individually. That, for instance, the logarithm of 365 might be found by some combination of the logarithms of 300, 60, and 5; and that the logarithms of all numbers whatever might be obtained by combinations of the logarithms of the nine figures; because, arithmetically speaking, all numbers are composed of those figures, each one, in a number consisting of several, being multiplied by a different power of 10, the modulus or root of the arithmetical scale.

If this could be done, it would reduce the construction of logarithms to a very simple process. But a very little consideration will show that this is impossible. For this purpose, let us take any number, 365 for instance, and analyze it according to its arithmetical composition, and see whether logarithms will apply to every process in that composition. 365 is $= 300 + 60 + 5$; 300 is $= 3 \times 10^2$, 60 is $= 6 \times 10^1$, and 5 is $= 5 \times 10^0$. So far as the composition of each of these three parts goes, the process is multiplication only, and therefore it could be performed by means of the logarithms; but when we look at the other part of the composition, and take the whole analysis,

$$3 \times 10^2 + 6 \times 10^1 + 5 \times 10^0,$$

we find that there are additions to be performed, and these are operations to which logarithms will not apply; consequently the scale of numbers gives us no assistance in the finding of logarithms, further than the place of the left-hand figure of the number determines the index. Consequently a distinct and separate operation is required for finding the logarithm of every prime number; but the logarithms of all composite numbers may be found by taking for each the sum of the logarithms of all its prime factors.

The explanations which we have given will, we trust, be found sufficient to convey an elementary notion at least of the nature of logarithms, and of the relation in which they stand to common numbers; and we shall close this section by adding a short explanation of the tables, and the notation of logarithms; and a brief recapitulation of the more useful logarithmic operations, the principles of most of which have been explained already.

TABLES.

In tables of common logarithms, there are usually two parts: the first containing the logarithms of all the integer numbers up to 100; and in this part of the table the numbers are placed in one column, and the whole of the logarithms, index and all, immediately against them in another column. The columns of numbers are marked N at top and bottom, and the columns of logarithms, Log. So that this table, as far as it goes, is perfectly simple. The next part of the tables contains all the numbers from 100 to 9,999, ranged in a column down the left of as many pages as are necessary; the number of pages depending, of course, on the size of the page and the type. The page contains eleven other columns; ten of which are occupied by the arithmetical figures, from 0 to 9 inclusive; and the last one, at the right hand, contains the differences of the logarithms, and is marked D at the top of the page. The column under 0 contains, against each number in the column of numbers, the decimal part only of its logarithm; and the other columns contain the logarithms of the same numbers, with the figure at the top of the column of logarithms included as a unit's figure after the three figures which are in the left-hand column of numbers at the beginning. Thus, if we were to look at the tables for the logarithm of any three figures, we would have only to find those figures in the column of numbers, and the decimal part of the logarithm would be against it in the first column of logarithms, under 0 at the top. If there were four figures, we could find them all, except the units in the left-hand column, as before; and then, if we followed the line of logarithms till we came to the column which had the units figure at the top, we would find the logarithm of the four figures there. If there were more than four figures, we could not find directly in the tables the logarithm of any more than the four nearest the left hand; but we could get an approximate value of the other ones by means of the differences which were marked in the last column of the page. These differences are those of the logarithms of the numbers consisting of four figures, and the next higher number of four figures; that is, a number which has the last of the four 1 greater. Now, any number of figures which can stand to the right of any one figure in a number, are always of less value than if that figure were 1 greater, and the places of the others supplied by 0s; and therefore if we take the difference, and multiply it by the figures which are in our number to the right of four, reckoning from the left, and cut off as many figures from the right of the product, as there are in the number by which we multiply the difference, the rest, or left-hand part of the product, will be a correction, which, added to the logarithm of the first four figures, will approximate the logarithm of the entire number. The approximation will be a very rude one, however, if there are more than one, or at any rate than two, figures; for every figure which is annexed to the right hand of the number, not only adds its own value as units, but multiplies the value of all the rest by 10. We mentioned already that as the numbers increased, the difference of the logarithms decrease much more rapidly; and therefore the connection very soon ceases to be accurate.

For convenience in reading, the table is divided by a black line across the page, at the end of every ten lines of figures; and the whole logarithm is written in each of the columns, in the first line of figures after this line of division; but in the other lines, if two or three figures at the beginning are the same in all the other nine lines, they are left out, it being understood that those which answer to the blank in the first line, are to fill it in the others. The use of the black lines is to guide one's eye more easily along in seeking the logarithm of four figures; and the blanks are merely contrivances of the printer to save a few types. If these directions are attended to, any one may find the common logarithm of numbers from the tables, without the slightest difficulty; and as for the indices, they are determined by the number of integer places, or of 0s at the beginning of a number wholly decimal, without any reference whatever to what the figures are, or what the decimal part of the logarithm, as found in the tables, is to be. The following numbers, which are expressed by the same figures, but of which, in consequence of the shifting of the decimal point, every one is but a tenth part of the one above it, will illustrate this:

Numbers.	Indices.
365000·	5·
36500·	4·
3650·	3·
365·	2·
36·5	1·
3·65	0·
·365	—1·
·0365	—2·
·00365	—3·
·000365	—4·

This requires no explanation; for by merely inspecting the column of numbers, and that of indices, and comparing them with each other, every thing about the index of the logarithm of any number may be readily understood, and we have only to bear in mind that — is the sign of division as applied to the numbers. In taking the logarithms of numbers from the tables, there is considerable advantage in the very simple operations of reading them in twos. Thus, if the logarithm is ·954291, it is much more easily remembered, if read "ninety-five, forty-two, ninety-one," than if we were to try to name all the figures singly, or the entire number as a whole.

The converse of this operation—that is, finding the natural number answering to any given logarithm—is just the opposite of finding the logarithm. Disregarding the index, we seek the tables for the logarithm; and, if we find it exactly, the number answering to it is the number, and the placing of the decimal point in it, or adding 0s to the right of it to make up the requisite number of integer places is determined by the index.

If the logarithm is not found exactly in the tables, we can, by reversing the correction, get a figure or two more to place to the right of those which answer to the logarithm. For this purpose, we take the logarithm in the tables which is next less than the given one, and get the difference between it and the given one for a dividend. Then, we place two 0s on the right of this dividend, and divide the whole by the difference of the logarithms, as marked in the right-hand column. It must be remembered, however, that the quotient of this division must make *two* figures, and if there is only one, an 0 must be placed before it in the number. The division might be continued to an interminable number of figures; but the labour would be useless, for none of them would be accurate except the first and second, and even the second would not be altogether correct in every case.

NOTATION.

The logarithm of any number, in a system having any modulus, may, if b is the modulus, and a the number, be simply indicated algebraically by the expression

$$b^n = a,$$

in which ^{n} denotes the logarithm, but does not express it in terms of the known numbers a and b. In order so to express it, it must be formed into a series, which will be interminable in all cases except those in which n is a whole number; and this series is the general formula, by substituting numbers for the letters in which, and performing the operations pointed out by the signs, the logarithm of any number may be found; but, as we already mentioned, the investigation of this is not purely elementary, and we are not yet prepared for entering upon it.

When a logarithm is expressed by a letter, a number, or any other single expression, simple or compound, it is pointed out to be a logarithm, by Log. or simply L. before it; and when it enters into a formula, or combination of quantities, we must attend to its nature, in order to know how it is to be affected by the signs. Now a logarithm is not only not of the same kind, but has actually no property in common with any quantity the value of which is immediately expressible by a common number; and therefore a logarithm and a quantity can neither have a sum nor a difference; so that the sign + or — between a logarithm and a quantity would have no meaning, or rather it would have a double meaning; for it would imply addition or subtraction as regarded the quantity, and multiplication or division as regarded the logarithm; and as no single operation can be both addition and multiplication, or subtraction and division, that which was indicated by such a sign would be impracticable in any other way than by finding the number answering to the logarithm, and dealing with it as a number—that is, adding it if the sign were +, and subtracting it if the sign were —.

A sign of multiplication between a logarithm and a quantity, means that the logarithm is to be raised to the power expressed by the quantity. But a sign of multiplication between two logarithms has not any real meaning; because there cannot be a product without a multiplicand; and the two logarithms are not only not multiplicands, but neither of them is even a multiplier, expressing the number of times that a certain number must be multiplied by itself, in order that the last product may be equal to another given number.

The sign of division between a number and a quantity points out that root of the logarithm, or the number for which it stands, which the quantity expresses; but if we were to consider the quantity as a dividend, the logarithm could not be a divisor; because a divisor must always be equal to some part of the dividend, and a logarithm is not equal to any part of a number.

But a sign of division between two logarithms has meaning, and implies that the logarithm which precedes it must be divided by the logarithm which comes after it; and, as is the case in every division of a quantity by another quantity of the same kind, the quotient is a number. Literally, it is the number of times which the one logarithm is contained in, or can be arithmetically subtracted from, the other. It is also, however, the converse of finding the power of a logarithm, because the power is found by multiplying the logarithm by the exponent; and as this power is the product of a logarithm and a number, either the one or the other of these may be found by dividing it by the remaining one. If we divide it by the number, we get the logarithm of the root; and if we divide it by the other factor, which is the logarithm of the root, we of course get the exponent of the power as a common number. Hence we obtain a means of applying logarithms to another problem, which could not be solved by the arithmetic of mere numbers:—

Given two numbers to find what power either of them is of the other one:—

The solution here evidently is: Divide the logarithm of the number which is considered as a power, by the logarithm of that which is considered as a root, and the quotient is the exponent in a common number.

If the question were, what power is a of b; then the expression, calling the unknown exponent x, would be $b^x = a$, $= \frac{\text{Log. } a}{\text{Log. } b}$, or Log. a ÷ Log. b. If the question were a numerical one, and it were demanded what power 8 is of 7; the only way to perform it would be to divide the logarithm of 8 by the logarithm of 7. From the tables these are,

$$\text{Log. } 7 = 0{\cdot}845098; \text{ and Log. } 8 = 0{\cdot}903090.$$

The last of which may be divided by the first, as follows:

```
0·845098) 0·903090 (1·044956
          0·845098
          --------
          379920 0
          338039 2
          --------
           41890 80
           33803 92
           --------
            8086 880
            7605 882
            --------
             480 9980
             422 5490
             --------
              58 44900
              50 70588
              --------
               7 74312
```

Thus we find that 8 is a power of 7, whose exponent is 1·044956, that is a very little greater than the number 1, which is evidently true; for 8 is a very little more than 1 time 7, which is the first power of 7, or 7^1.

In the operation for finding it, half the figures, namely those which stand under the 0s annexed to the remainders, might have been dispensed with; and we have marked them off by leaving spaces between them and the others. The reason that these figures could be dispensed with is, that they are uncertain; because neither the divisor nor the dividend is terminate, or could terminate if we extend it ever so far; and therefore both of them should have been continued in the figures of the logarithm, in order to be perfectly accurate. We do not, however, know those figures beyond the six which are in the tables; and thus all that we can obtain is an approximation; and if, at every step of the division, we had, instead of annexing 0s to the remainder, omitted the same number of figures on the right of the divisor, in succession, as there are 0s used at every step, and only taken in with the product the number that had to be carried, our operation would have consisted only of the figures to the left of the blank space, and, excepting perhaps a single 1 in the last figure, our quotient would have been the same. This, by the way, is a very convenient mode of abridging the division of decimal numbers in all cases where 0s have to be added to the successive remainders; but it is so simple that it requires no further explanation.

This number 1·044956, which we have obtained, is not quite true in the last figure, and probably not in the last except one; but we can depend upon it to four decimal places, which gives us our exponent to the nearest ten thousandth part of exponent 1; and simple as the operation appears, were it not for the help of the logarithms, we should have had very serious labour in finding it out. This will readily appear when we consider that we cannot multiply any number a fraction of a time, in any other way than by raising it to the power of the numerator, and extracting the root of the denominator, and as this exponent, expressed fractionally to the fourth figure of the decimal, is $\frac{10442}{10000}$; we should have had to work by powers and roots having these high exponents, and our lines of figures would have been several feet or yards in length.

And even this would not have been the tedious part of the matter; for these very high exponents are not data given us, they are the results which we are to find, and we must find them both at the same time; so that the finding of them may be said to be wholly beyond the power of common arithmetic, whereas, by the help of logarithms, the operation is reduced to the performing of one simple division.

The number which we have thus obtained, is not a common number, but an exponent; and therefore we could not use it arithmetically, unless we were to find the natural number which answers to it. We could, however, use it as a logarithm, only it does not belong to our common tables, as it is the logarithm of 8 taken in terms of the number 7 as the root of a system, the same as 10 is the root of our common system.

We believe there is only one other little matter necessary to be mentioned in logarithmic notation, and it is a mere matter of form. Negative indices are sometimes expressed by writing — over the figures instead of before them. Thus $\overline{3}\cdot$ instead of — 3·. They are also sometimes expressed by means of their complements to 10. Thus — 3· is expressed by 7·. No advantage is, however, gained by this means; for, in adding, we must subtract as many 10s from the sum as there are complements; and in subtracting we must add as many times.

LOGARITHMIC OPERATIONS.

For the reasons already explained, no addition or subtraction of numbers can be expressed by logarithms; and the sum or difference of a number and a logarithm does not express any thing which has a meaning or could exist. In stating this, we are to understand that the word number is used as a short expression for all common numbers, and for all quantities that are capable of being, in whole or in part, expressed by common numbers, without in any way altering their nature.

Multiplication is performed by adding the logarithm of the factors; and *Division* by subtracting the logarithm of the divisor from that of the dividend. If a number of successive multiplications and divisions, without any intervening additions or subtractions, have to be performed, all the logarithms of the factors may be collected into one sum, and the logarithms of all the divisors into another; and when the last of these is subtracted from the first, the remainder is the logarithm of the ultimate quotient, or of the value of the expression in a single number.

In performing the subtractions, it must not be lost sight of that they are really divisions; and that though the logarithm which we have to subtract be really greater, both in the index and in the decimal part, than the logarithm from which we have to subtract it, there will still be a real and positive remainder; whereas if, in common arithmetical subtraction, we try to take a greater number from a less, we should find ourselves *minus* the difference; that is to say, the numerical difference would be wholly affected by the sign —, and really be a quantity less than nothing. But in division we never can have a negative quotient, if the divisor and dividend have the same sign; and as the decimal part of every logarithm is + or positive, there is always something positive to divide, as well as something positive to divide it by, whatsoever may be the signs of the indices in the case of logarithms.

Involution is performed logarithmically, by multiplying the logarithm of the root by the exponent of the power; and the natural number answering to the product is the power required, as a natural number: and *Evolution* is performed by dividing the logarithm of the power by the exponent of the root; and the natural number answering to the quotient is the root itself.

In multiplying a logarithm with a negative index, the product which arises from multiplying the left-hand figure of the decimal part, if it contains any thing which has to be carried to the product of the negative index, must, though positive itself, send the quantity carried to the index as negative. This may at first sight seem a little singular; but it is nevertheless strictly true. The index affects the whole value of the number for which the logarithm stands, in as far as it determines its place in the scale of numbers; and thus, though it makes no alteration in the figures of which the logarithm is composed, it affects the whole value of that logarithm; for if it did not, the logarithm would not be a faithful representative of the number. Therefore, when the decimal or positive part of the logarithm, having a negative index, is so multiplied as that there is something to carry from the left-hand figure to the index, the number which is thus carried becomes of itself an index by the transfer, and acquires the negative sign by passing the decimal point, just as a number which is passed the decimal point into the decimals requires — before its exponent.

On the other hand, in dividing logarithms with negative indices, the quotient, if it amount to 1 or more, must have the sign —; but whatever remainder may be over on dividing the negative index, will go over to the first figure of the decimals as a positive quantity, every 1 in it counting as 10, as in every other case in division.

Such are the leading principles in the arithmetic of exponents, and the nature and use of common logarithms; and as there is no contrivance in the whole compass of the mathematical sciences so well calculated for abridging labour, or enabling us to perform operations which we could not perform by common arithmetic, this is a portion of the subject which ought to be fully understood by every one who wishes to be even a tolerable accountant; while those who aim at any of the practical connexions between arithmetic and geometry, will find most untoward work of it if they do not learn the ready use of this powerful instrument. It is true that the tables are easily understood, and that the operations are very simple; and that these may be committed to memory, and put in practice without any understanding, just as a parrot learns to repeat a sentence, or whistle a tune; but those who are thus unfortunate, never know when they are right; and therefore when the slightest trifle arises which is not found in the formula, they are completely at a loss, and sure to be wrong.

SECTION XV.

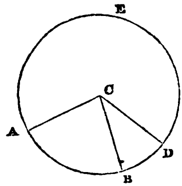
INTERSECTIONS OF LINES AND CIRCLES.

THE circle is the foundation of all our purely geometrical notions of the equality of lines and angles; and these are the elements of which all our other and more complex notions of geometrical equality are formed; therefore, after we have once fully understood the nature of lines and angles, and the general doctrine of relations, of which an outline has been attempted in the former sections of this volume, our next object should be to make ourselves well acquainted with the uses of the circle, in enabling us to compare the lengths of lines, and the magnitudes of angles. This we shall attempt in the present section.

It will be borne in mind that all radii of the same circle, or of equal circles, are necessarily equal; which follows from the way in which a circle is described, and is our primary notion of it, or that upon which the definition is founded. This, therefore, is a simple cause of the equality of lines, which does not need or admit of any proof.

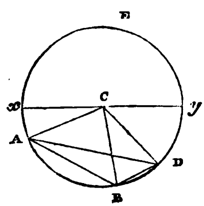
It will also be borne in mind, that the circumference of a circle is the measure of all the angular space round a point; that is, it is equal to four right angles; and that any portion of the circumference of any circle, and the same portion of four right angles, are naturally and reciprocally the measures of each other; consequently the larger the portion of the circumference, the larger is the angle; and conversely, the smaller the portion of the circle, the smaller is the angle. It must be understood, however, that it is not the absolute length of the arc in any known measure, but its length as compared with that of the whole circumference of which it is a part, which determines the value of the angle.

When it is not otherwise mentioned, the centre of the circle is the point to which the angle is referred, because it is the only point within the circle which stands in the same relation to every point of the circumference. In the language of geometry, the arc is said to *subtend* the angle, that is, to "hold under it;" so as to tie it to one definite value; and the angle is said to *stand upon* the arc. Thus if A B D E is any circle whatsoever,



of which C is the centre; and the portions of the circumference from A to B, and from B to C, any two arcs. Draw the radii A C, B C, and D C, from the extremities of the arcs, to C the centre, and the angle A C B, and arc A B are reciprocally the the measures of each other, and so are the angle B C D and the arc B D. Also the angle A C D, which is the sum of the two angles A C B and B C D, and the arc A D, which is the sum of the two arcs A B and B D, are reciprocally the measures of each other. If the arc A B is greater than the arc B D, the angle A C B is greater than the angle B C D; if equal, equal; and if less, less. Further, if the one arc is any multiple or part of the other arc, the angle standing on or subtended by the first arc, is the same multiple or part of the angle standing on or subtended by the second arc. Arcs, and the angles which stand on them, are therefore proportional quantities.

But if we, as in the following figure,



draw the straight lines A B and B D, joining the extremities of the two arcs A B and B D, and also the line A D joining the extremities of the arc A D, which is the sum of the other two, these lines subtend their respective angles at C, or tie them down to the same definite values as they are tied to by the arcs.

These lines are called the *chords* of the arcs, and they subtend both the arcs and the angles; so that by means of them we are enabled to get angles expressed, or compared with each other, in terms of straight lines; and this is one important step towards bringing geometry within the reach of common arithmetic.

From a mere inspection of this figure, it will be seen, that each of the chords, A B, B D, A D, is the chord of two arcs, the one less than a semicircle and the other greater; so that, in these cases, and in every possible case, the two together are equal to, or make up a complete circumference. Every chord, too, subtends two angles, or fixes the positions of two radii of the circle, which divide the angular space round a point into two parts, which parts, taken together, make exactly four right angles.

If the chord is a diameter, or passes through the centre, each arc is a semicircle, and there is no angle at the centre, as happens in the case of the diameter *x y* in the above figure. In every other case, there is a salient angle which is less than two right angles, and a re-entering angle, which is just as much greater. The greater the chord the greater the salient angle, and also the arc which subtends that angle; but no arc subtending a salient angle can be so great as a semicircle; and therefore the chord subtending no salient angle can be so great as the diameter.

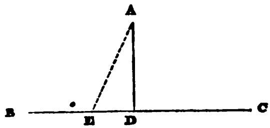
But though it is thus evident from the simplest property of the circle, that the chord of an arc increases as the arc increases, yet this holds only till the greatest possible chord, the diameter, is arrived at; for, beyond this, the arc is greater than a semicircle, the angle is a re-entering angle, and the chord becomes that of a salient angle directed the other way, and subtended by an arc as much less than a semicircle, as the arc subtending the re-entering angle is greater.

Whatever the diameter of the circle is, therefore, the value of the chord begins at 0, increases to a maximum at the semicircle, where it is equal to the diameter; and diminishes from that maximum till it is again 0. Hence the rate of its increase must diminish from 0 to the maximum, and the rate of its diminution must increase from the maximum to 0,—the one being merely the reverse of the other.

In consequence of this, the chords of arcs cannot have the same ratios to each other in the measure of lines, that the arcs themselves have in degrees of the circumference. We do not in the meantime, however, require to investigate the law of their variation; for it is evident, that, up to a semicircle, the greater arc has the greater chord; and as every salient angle is less than the measure of a semicircle, or 180°, we have this general conclusion: The greater salient angle is subtended by the greater line, and conversely. But in a triangle every angle is subtended by the opposite side. Therefore, in every triangle, the greater angle is opposite the greater side, and conversely. We shall presently see that this is a very important principle.

1. The shortest distance between a point and a straight line, is the perpendicular drawn from that point to the line.

Let A be any point, and B C any straight line; the shortest distance from A to B C is the perpendicular A D, which makes the angles A D B and A D C equal, and each a right angle. If not, take any point E, in B C, upon either side of D; join A E, and let A E, if possible, be less than A D.



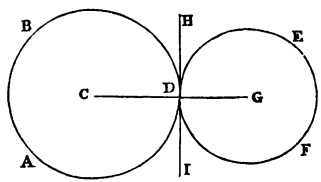
A D E is a triangle, of which the three angles together = two right angles; and the angle at D is a right angle, and equal to both the angles at A and E. Consequently, the angle D is greater than the angle E, and the side A E opposite the greater angle, is greater than the side A D, opposite the less. But E is any point, and wherever it were taken, in B D or D C, A D would still be less than A E. Wherefore, the perpendicular is the shortest distance.

2. If two angles of a triangle are equal, the sides opposite those angles are equal; and conversely, if two sides are equal, the angles opposite them are equal. This follows, from the greater side and greater angle being opposite; for though the equal sides, as lines, have not the same ratio as the opposite angles in circular measure, yet equal ratios are of course equal. Hence also, if the three sides of a triangle are equal, the three angles must also be equal, and conversely.

3. If a straight line touches a circle, but does not cut it, that is, does not pass within the circumference, then the radius, or line drawn through the centre of the circle to the point of contact, is perpendicular, or at right angles, to the touching line, or *tangent*, as it is usually called. For, the point of contact is in the circumference, and every other point in the touching line is without the circumference, and therefore more distant from the centre than the point of contact. But the perpendicular is the shortest distance; and therefore the line drawn through the centre to the point of contact, is the perpendicular to the tangent, or touching line, at that point.

4. If two circles touch each other externally, the line joining their centres passes through the point of contact; the portions of it between their centres, is equal to the sum of their radii; and a line crossing this at right angles in the point of contact, touches both circles, but cuts neither.

If the circles A B D, of which C is the centre, and D E F, of which G is the centre, touch each other externally in the point D, the line C G, passing through C and G, the centres, passes through the point of contact; C G is the sum of the radii of the circles; and H I, crossing C G in the point D, touches both circles, but cuts neither.



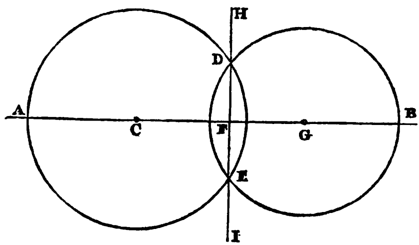
C D and G D are the shortest distances from C and G to D; C G is the sum of C D and G D; and every other point in H I is further from both C and G than the point D is.

5. Upon the very same principle it follows, that if one circle touch another internally, the straight line at right angles to the tangent, at the point of contact, passes through the centres of them both.

6. Also, if the distance of the centres is greater than the sum of both radii, the circles will not touch each other; and if the distance of the centres is less than the sum of the radii, they must cut each other.

In this case, as the line joining the centres, if continued far enough both ways, divides both circles exactly in the middle, or into two semicircles, which are every way equal and equally related to each other, though they are symmetrical magnitudes, or the one turned the one way and the other the opposite way, it follows, that whatever can be said of the segment or arc of each, between the line joining the centres and the point of section, can be said of the corresponding or symmetrical segment on the other circle. A diagram will illustrate this more clearly than words.

Let A B be a line passing through C and G, the centres of any two circles, whose distance from each other is less than the sum of the radii of the circles.



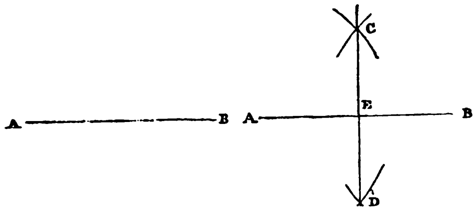
Then the circles must cut each other in some two points, D on the one side of the line, and E on the other, and all the segment or arc of each circle which lies between these points of section, must be within the circumference of the other circle. But the parts of each of the two circles which are above the line A B, are equal to, and symmetrical with, those which are below the same; therefore the point of section at D stands in exactly the same relation to the centres C and G and the line which passes through them, as the point of section E does. Through the points D and E, draw the cross line H I, and H I must cut A B at right angles in the point F; for whatever can be affirmed of the angle A F H on the one side of A B, can be affirmed of the angle A F I on the other side; and in like manner, whatever can be affirmed of the angle B F H, can be equally affirmed of the angle A F H. Wherefore H I is at right angles to A B; and the part D E, between the intersection of the circles on the side toward H, is equal to the equally and symmetrically intercepted part F E, on the under circle of A B.

From this we can immediately obtain the means of performing two practical operations or problems: first, to divide a given straight line into two equal parts, by another line crossing it at right angles; and secondly, to construct a triangle with three given straight lines,—but any two of these must be together longer than the third one.

To divide a given straight line into two equal parts, or as it is usually termed, to "bisect" a straight line. Let A B, in the following diagram, be the line which it is proposed to divide into two equal parts.

Upon the extremity A as a centre, and with any radius greater than half of A B, describe an arc of a circle on each side the line; and upon the point B, and with the same radius, describe arcs of an equal circle, cutting the former in the points C and D; draw a line through C and D, cutting A B in the point E; then A B is bisected, or divided into equal parts in E, and C D is at right angles to A B.

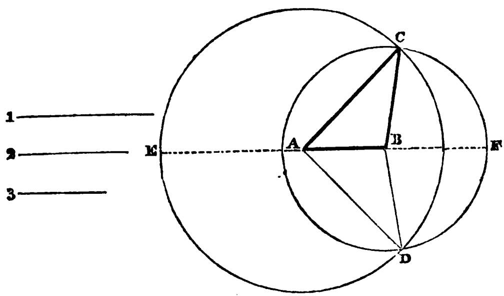
In performing this problem, it is not necessary to draw the whole of the two equal circles, but merely a part of each, so that the one may cut, or cross the other in the two points C and D, on the opposite sides of the line, as the places of these two points are all that is required, in order that C D may be drawn in the position required. Nor have we thought it necessary to prove the equality of A B as it stands singly and as divided, by showing that they are both radii of the same circle. There are few practical cases in which this can be done; and therefore the equal measure of the two is the most useful standard of equality. The length of A B, where it appears as a single line, is taken between the points of a pair of compasses, or by any other means that will take and preserve it exactly; and the ex-



treme points of this being marked at A and B in the second figure, the line joining them is of the same length as the original A B. So also, in describing the arcs of equal circles from A and B as centres, the compasses, with the distance between their points unaltered, are carried from the one to the other, which is all the equality of radii which we can obtain in practice, and, consequently, any theoretical nicety beyond this is mere pretension.

The bisection of A B by C D in the point E, and also the fact of the one line being at right angles to the other, follow necessarily from what was said in article 6. It was there shown, that when two circles cut each other, the line joining the points of section must be at right angles to the line passing through, or joining the centres of the two circles; and because the radii of the two circles are equal in the present case, it follows, that the point of intersection of the lines, E, has the same relation to each of the two centres, A and B; and as the only relation that it has, or can have, to those centres, is its distance from them (for a point in a straight line can have no other relation to another point in the same, than the distance they are from each other,) its distance from each must be the same, that is, the line A B is bisected on the point E.

The second problem resulting from this is, the construction of a triangle, the sides of which shall be respectively equal to three given straight lines; but any two of these lines must be together greater than the third, otherwise there can be no triangle. The principle in this is not very different from that in the former; but the practical application of the principle is different. One of the given lines, no matter which, must be made the distance of the centres of two circles, whose radii are the other two lines. These centres will be the points of two angles of the triangle, and the point of the third angle will be the intersection of the circles. But as there are always two points of intersection, when two complete circles cut each other, two equal and symmetrical triangles, that is, triangles turned opposite ways, may, in every case, be constructed of the same three straight lines. We shall take an example, as in the following figure.



Let 1, 2, 3, be three straight lines, it is required to construct a triangle whose sides shall be respectively equal to 1, 2, and 3.

Make A B equal to any of the given lines, for instance, to 3. Then, about A as a centre, and with a radius equal to another of the given lines, 1 for instance, describe a circle, E C D; and about B as a centre, with a radius equal to the remaining line 2, describe a circle C F D; and, because the lines 1 and 2 are, by hypothesis (or that which was assumed) together greater than 3, they must cut each other in some two points C and D, one on the one side of A B, and the other on the opposite side. Join A C, B C, and A D, B D; and A B C, A B D, are two equal and symmetrical triangles, each having its three sides respectively equal to the given lines 1, 2 and 3,—namely A C in the one, or A D in the other, equal to 1; B C in the one, or B D in the other, equal to 2; and A B, which is common to both triangles, equal to 3.

Both triangles would have been the same in size and shape, although either of the other two given lines had been taken as the one, the extremities of which were to be the centres from which circles were to be described, with the remaining lines as radii; but the side on which the two triangles applied to each other would have been different, and so would have been the positions of the other parts.

This fact of the symmetrical triangles being constructed by the same operation, is a very important one; for it enables us to assume that symmetrical surfaces, or those that have all their sides and angles equal, but placed in a contrary order, are equal; and this once admitted, saves much of that indirect demonstration, which, though perhaps strictly geometrical, is far more tedious, not more satisfactory to the mind, and certainly far less useful.

This seems to be one reason why geometry is so completely a sealed book to the greater part of mankind, not only to the ignorant and the unthinking, to whom, of course, all the sciences and all subjects of reasoning are sealed books, but to the great majority of those who are educated, and who, in the years of their pupilage, have conned over the Elements of Geometry, among other subjects, which were worked at only to be forgotten, or rather, which could not be forgotten—never having been understood.

The particular way in which anything stands, lies, or is situated, does not affect the value of that thing. A man is in no way a different being when he stands with his face to the north, than when he stands with it to the south; a sovereign is the very same coin, whether it is in the right hand pocket or the left; and when it is put down upon a counter in payment, the shopkeeper makes no dispute about the side that happens to be uppermost, so that the coin is of sterling metal and the proper weight, which are the true elements of its value. In common life, we do not make position an element of value, except of that particular value which arises from the fact of the thing valued being in a particular situation; and this has nothing to do with the abstract or intrinsic value of the thing. A sovereign in the pocket is a very different thing to a hungry man in a city, where there is food ready dressed within a few yards of him, with a vender seeking customers, to what it is on a barren hill twenty miles from any human habitation, or on a barren rock in the middle of a wide ocean; but in all these situations the intrinsic value of the sovereign is exactly the same, and the difference of its value to the man depends upon circumstances wholly external and independent of the value of the coin itself.

It is the abstract values of things—their values in themselves and without any regard to external circumstances—which are the subjects of all the mathematical sciences; and as geometry is the branch which applies to magnitude, magnitude in the abstract, and without any regard to circumstances external of the magnitude under consideration in the particular case, is the proper subject of geometry.

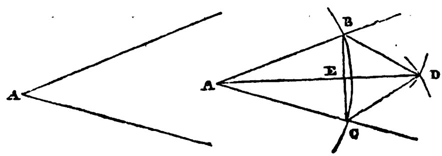
In the above figure, the triangle A D B is as equal in the whole, and in all its parts, to the triangle A C B, as if they were both turned the same way. The sides A D and A C are radii of the same circle, and so are B D and B C; while A B, being the same identical line in both triangles, cannot admit of dispute. But, if in the same triangle, equal sides have equal angles opposite to them, the same must hold in the case of triangles which are every way equal. In the case under consideration, the two angles at A must be equal to each other, and so must the two angles at B, and the angle at C in the one triangle must be equal to the angle at D in the other; for the equal ones are not only opposite to, but contained by, equal sides in the two triangles.

It is true that the equality of these two triangles, A C B and A D B, are equal by construction and not by measurement; but still the equality is not on that account the less true, or the less evident. If we suppose the line A B to be continued, as it is in the dotted line to E in the one circle, and to F in the other, then both circles are equally though symmetrically divided by this line, as it passes through both their centres; and therefore whatever is true of them on their intersection on the one side of this line, must be equally true of them on the other side, the difference being merely in position and not in form or value.

We have treated this principle of what may be called "symmetrical equality," at some length; because, though like the principle of motion, it is not formally admitted into the elements of geometry, yet, like that principle, it must be afterwards assumed. Thus the equality of the two parts into which a circle is divided by a diameter or line passing through the centre, is neither taken as a definition, nor demonstrated as a theorem, it is barely assumed. The two semi-circles are symmetrical; and thus the modes of proof generally admitted in the elements, do not apply to them. But surely it is better at once to explain the general principle, than tacitly to assume the particular case. It is not in geometry only that this unwise mode of dealing with a subject is a stumbling-block in the way both of knowing and of doing. We shall now show how this principle may be applied in one or two cases.

7. To divide an angle into two equal parts, or, as it is usually expressed, "to bisect a given rectilineal angle."

On the angular point A as a centre, and with any radius not longer than the lines which contain the angle, describe an arc cutting those lines in the points B and C. Then from B and C, as centres, and with any radius, only it must be the same in both, describe two arcs to the other hand from the angular point, crossing each other in the point D. Draw a line through D and the angular point, and this line A D bisects the angle.



Draw B D and C D. Then A B D and A C D are symmetrical triangles, having A B = A C, B D = C D, and A D common to both. Therefore they are every way equal, namely, the two angles at A are equal, or A D bisects the given angle at A; also the angle at B = the angle at C; and the two angles at D are equal to each other; for all the angles of which equality is alleged are opposite to equal sides of the equal symmetrical triangles.

8. If we examine the last part of this figure a little more closely, we shall find that it has further information to give us. Draw B C, and it is the chord of the arc which cuts the sides of the angle in the points B and C. A D bisects the arc, because it bisects the angle which that arc subtends at the centre A; and if it bisects the arc, it must bisect the chord of that arc.

Hence B E is equal to E C; and the angle A E B is equal to the angle A E C; for they are opposite equal sides in the equal and symmetrical triangles. A E B and A E C are on opposite sides of A E. But they are the angles made on one side of a line by another crossing it, and therefore they are together equal to two right angles; and they are equal to each other; therefore each of them is a right angle. Consequently A D bisects the line B C at right angles in the point E.

But A D is a line passing through A, the centre of the circle of which the arc B D is part; B C is a line meeting the arc in the two points B and C; therefore,

9. If a line passing through the centre of a circle bisects another line which does not pass through the centre, but meets the circumference both ways, it cuts it at right angles; and if it cuts it at right angles, it bisects it. Consequently, if one line bisects another at right angles in a circle, the line which bisects the other passes through the centre of the circle, or would pass through it if continued far enough.

Also, if two lines bisect other lines at right angles in a circle, both the lines which bisect the others must pass through the centre of the circle, that is, the centre must be in the point where they cross each other, for that is the only point common to both lines.

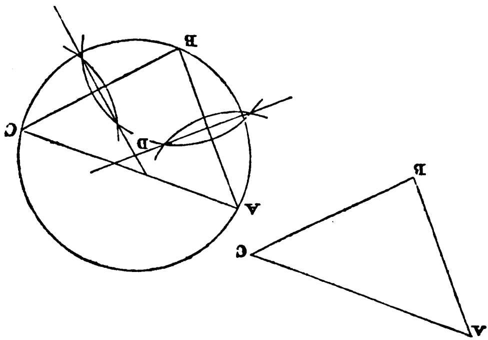
This principle enables us to perform two problems:

First. To find the centre of a circle. Draw any two chords and bisect them at right angles by two lines, and the intersection of these lines is the centre of the circle. These two chords may have one extremity at each and the same point,—hence, we have this truth:

If three points are given, the circle passing through those points is also given; and the construction of this circle from the data is our other problem.

Secondly. To describe a circle through three given points; or, which is evidently the same, to describe a circle which shall touch all the three angles of a given triangle. Bisect two sides of the triangle at right angles, and the intersection of the bisecting lines is the centre of the circle. We shall illustrate this by a diagram, as it is often useful in practice.

A B C is any triangle, it is required to describe a circle which shall touch all its angles.

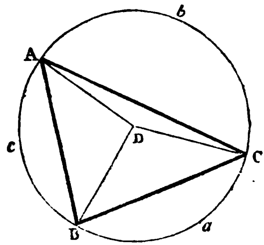


Bisect A B and B C, by describing arcs on the angles as centres, with radii larger than half the side, and the same for both ends of each; draw the bisecting lines through both crossings of the arcs on each, and produce them till they cross each other in D; and D is the centre of the circle. The radius is the distance of D from any of the angles.

There is something to be learned from this figure, and so we shall repeat it, leaving out the arcs and lines which were necessary for finding the centre D, drawing lines from that centre to the angles, and strengthening the sides of the triangle.

An angle is said to be at the circumference of a circle when its vertex or point is in that circumference. Thus, the angles A, B, and C of the triangle in the following figure are all at the circumference.

An angle at the circumference is said to stand upon the arc which is intercepted between the opposite ends of the lines that form the angle, and to be contained in the remainder of the



circumference. Thus the angle A stands on the arc *a* (extending from B to C), the angle B stands on the arc *b* (extending from C to A), and the angle C stands on the arc *c* (extending from A to B).

Also, the angle A is contained in the sum of the two arcs *b* and *c*; the angle B in the sum of the two arcs *a* and *c*; and the angle C is contained in the sum of the two arcs *a* and *b*; therefore

The three angles A B and C, taken altogether, stand on the whole circumference of the circle. But A B and C are the three angles of a triangle, and as such they are equal to two right angles. Wherefore,

The sum of all the angles at the circumference of a circle, which stand on the whole of that circumference, is equal to two right angles.

But all the angles at the centre of a circle standing on the whole circle, are together equal to four right angles, for they are all the angles round a point. Therefore,

All the angles at the centre standing on the whole circumference, are together double all the angles at the circumference standing on the same.

But the angles at the centre are in proportion to the arcs on which they stand, for the arcs are their measures, and measures are the foundations of all proportion. Therefore, again,

Any angle at the centre is double the angle at the circumference, which stands on the same arc.

Refer again to the diagram. The lines A D, B D, and C D, are drawn from the angles A B and C, to D the centre of the circle; and they stand on the same arcs as the angles of the triangle. A B C and A D C both stand on the arc *b*; A C B and A D B on the arc *c*; and B A C and B D C on the arc *a*. Therefore A D C is double A B C; A D B is double A C B; and B D C is double B A C

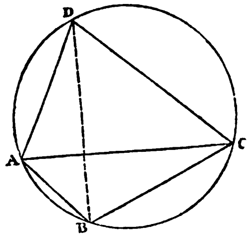
The fact of the angle at the centre being double the angle at the circumference, might have been arrived at by applying the principle of symmetrical triangles to a particular case; but the general investigation is much more satisfactory.

10. It will be seen that the magnitude of the angle at the circumference increases with the arc or segment upon which it stands, and not with that in which it is contained. It is always half the angle at the centre; and as the angle at the centre is in proportion to the arc on which it stands, the angle at the circumference, which is the half of it, must always be in proportion to half that arc.

The angle in a segment is, therefore, always inversely as the segment which contains it, and directly as the segment upon which it stands. The words "segment" and "arc" being synonymous in this use of them.

These two segments, namely, the one that contains the angle, and the one upon which the angle stands, and which determines the magnitude of the angle, always between them make up the whole circumference. But the angles standing on the whole circumference, and having their vertices at the circumference, are always equal to two right angles. Therefore, if the circumference of a circle is divided into any two segments, the angles in those segments are always together equal to two right angles, as they are the supplements of each other.

Thus, if any circle, A B C D, is divided into two segments by

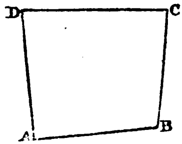


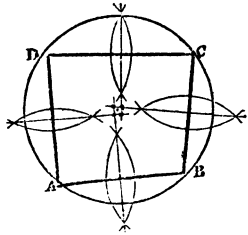
any chord, as for instance the chord A C, the angles A B C in the one segment, and A D C in the other, are always equal to two right angles in which way soever the chord may divide the the circle. Also, if another chord is drawn, joining the vertices of the angles, as the dotted line D B, the angles D A B, D C B, are together equal to two right angles. Hence, we have this general conclusion:

If a four-sided figure can be inscribed in a circle, that is, if a circle can be drawn so as to touch all its angles, the opposite angles of that figure are together equal to two right angles.

But this is not a property of all four-sided figures, for there is a condition in it, and we must have the means of knowing whether any given figure has or has not this condition; and this takes the form of the following problem:

To determine whether a given four-sided figure can or cannot be inscribed in a circle. As all the four sides are here placed in a circle, the line which bisects each of them at right angles must pass through the centre, and unless these lines all meet in one point, the figure cannot be inscribed in a circle. We shall take a figure at random:



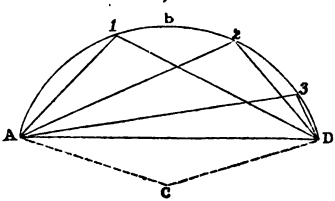


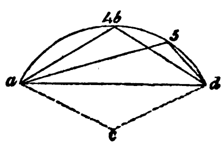
The first of these is the figure made at random; the second is the trial, by bisecting the sides at right angles, as formerly explained; and as there are four crossings as marked by the dots, the figure cannot be exactly inscribed in a circle; but it can be so very nearly, as the dots are close together; and a circle drawn from the fifth dot in the centre touches the angles B and C, but it is a little within at A, and a little without at D.

Another truth which follows from the same principles is the following:

All angles in the same segment of a circle are equal to each other, in what places soever their vertices may be situated, and whether the segment be part of a larger circle or a smaller.

Thus if A B D and *a b d* be equal segments, that is, each containing the same part of a circumference, but the one part of a larger circle than the other; then the angles which have their vertices at 1, 2 and 3 in the first, and at 4 and 5 in the





second, and also all possible angles that could be made in either segment or in any other equal segment whatever, are exactly the same portions of a circumference. For,

Find C and *c*, the centres of the segments, and draw the dotted lines A C, C D in the one, and *a c*,*c d* in the other; and as the segments are the same parts of the circumference, the angles A C D, and *a c d* are equal, and so are the re-entering angles on the opposite sides of the dotted lines, which are their supplements to four right angles. But these supplements are double of any of the angles in the segments, for they are the angles at the centres answering to the arcs on which the supplements stand. Therefore, all the angles are halves of equal quantities; consequently the angles themselves are all equal.

Any angle in a semicircle is a right angle, because it is equal to the angle in the opposite semi-circle, and the angles in the two segments which make up an entire circumference are always together equal to two right angles.

For the same reason, an angle in a segment less than a semicircle is always greater than a right angle; and an angle in a segment greater than a semicircle is always less than a right angle. The difference of the angle from a right angle is always half the difference of the arc from a semicircle; but it is less in the case of greater, and greater in the case of less.

11. We are now prepared to solve the following problem:

Upon a given straight line, to describe a segment of a circle, which shall contain an angle equal to a given angle.

This problem is often of much service, to those who are handy with a pair of compasses, in the construction of plans; but before we proceed to it, we must premise another:

To make, at a point in a given line, an angle equal to a given angle.

This is nearly self-evident: let A be the angle, and B the point in the line. On A describe an arc C D, and with the same radius describe an arc from B as a centre and on the side of the line, and toward that hand where the opening of the angle is to be. Then with a radius equal to the chord of C D, and from the point where the second arc meets the line, describe another arc cutting that one; draw a line through the point B and the intersection of the arcs, and the angle is made.

Draw the chords C D and E F, and they are equal, being chords of equal arcs; and the other sides of the triangles are all made equal. Therefore, the angle at B is equal to the angle at A.

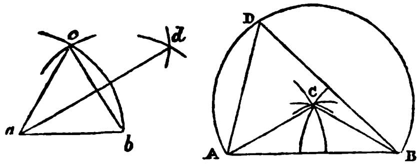
Let us now return to our main problem: given any line

A ———————————— B

it is required to construct upon it a segment of a circle that shall contain an angle equal to a sixth part of a circumference, or 60°. This is the angle of an equilateral triangle, for it is the third part of two right angles. The segment which we want must be greater than a semicircle, because the angle in it is less than a right angle. The given line A B is the chord of the arc on which the segment has to stand; and therefore it must subtend at the centre 120°, which is double the angle in the segment.

The problem is thus reduced to this: To apply to the given line A B a triangle which shall have its other two sides equal to each other, and the angle between them equal to 120°, or one third of a circumference; and the point of this angle will be the centre of the circle. The segment required will thus be two-thirds of a circumference.

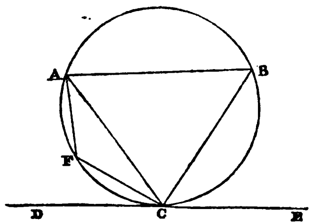
We have not yet mentioned the method of constructing a triangle from any data except the three sides: but the present is a simple one, and we have data enough. The lengths of the equal sides we cannot know; but we know the sum of the angles opposite them; for it is 180°—120°, that is 60°; and they are equal, so that each of them is 30°, or half the angle of an equilateral triangle.



Draw any line *a b*, and from the ends as centres with a radius = *a b*, describe arcs, crossing in *c*; join *c* to *a* and to *b*, and the triangle is equilateral by construction. On *b* and *c*, with the same radius, describe arcs crossing in *d*, and *d a b* is half the angle of an equilateral triangle. Apply this angle at each extremity of A B, and produce the line till they meet in C, and C is the centre of the segment. On C, with C A or C B for radius, describe the segment A D B, and the angle formed by the lines D A and D B, or any other formed by lines from A and B meeting at any point in the segment, is an angle of 60°.

12. If a straight line touch a circle, and if from the point of contact another straight line is drawn cutting the circle, the angles which this makes with the touching line or tangent, are equal to those in the alternate segments, that is, the segments on the opposite sides of the line which cuts.

Let D E touch the circle A B C, and let C A, drawn from the point of contact, cut the circle in the point C; then the angle



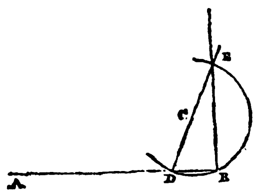
A C E on the right of A C is equal to the angle A F C on the left hand segment; and the angle A C D on the left of the line is equal to the angle A B C in the right-hand segment.

This still depends on the same principle. The angles on the opposite side of the line which cuts, are the supplements of each other to two right angles; and the angles in the opposite segments are the supplements of each other also; only the angles in the segments are inversely as the segments, while the segments themselves are directly as the angles made by the cutting line and the tangent.

If the line which cuts passes through the centre, it must make right angles with the tangent, and divide the circle into two semicircles, the angles in which will also be both right angles. The angles of the lines and the angles in the segment are thus both equal in the same position of the line which cuts, namely, when it is perpendicular. But when the line slopes, the angle and the segment from which it slopes both increase, and the angle in the segment diminishes at the same rate; and just as much as the angle and the segment increase on the side from which the line slopes, they diminish on that toward which it slopes, and as the segment diminishes the angle in it increases at exactly the same rate. Therefore, in every position of the cutting line, the angle made with the tangent is equal to that in the alternate segment.

13. At the extremity of a given line, to draw another line which shall be perpendicular to the given one. This may be done upon the principle that the angle in a semicircle is a right angle.

Let A B be the line, and the perpendicular to be drawn upwards from the extremity B. Take any point C, near but not at the end of the line, and at some distance from it on the same side to which the perpendicular is to be drawn. From C as a centre, and with a radius extending to B, but not beyond it, describe an arc, cutting the line in D, and touching the extremity B, and continue this arc till it is more than a semicircle. Then from D, through C, draw a line cutting the arc in E; and a line from B passing through E is perpendicular to the line A B. For, since D E passes through the centre C, the arc D B E is a semicircle; and consequently the angle D B E is a right angle.



A perpendicular to a line at any other point of its length could evidently be drawn in the same manner, the point being made to answer to B in the above figure. When, however, the point is not near the end of the line, the usual way is to take equal distances on each side of the point, and proceed as formerly directed for bisecting a line at right angles; so that we need not introduce a figure.

The first of these operations does not admit of being put in the inverse method; for, from a point without a line to let fall a perpendicular on the extremity of the line, is not a problem, if the position of the line is fixed and determinate. There is only one situation of the point in the direction parallel to the line that will answer in this case; and, so if the operation is possible, the solution is involved in the data, and there remains nothing to do but to draw the line through the two given points, which determine its direction.

If, however, the position of the line to which the other is to be perpendicular is not fixed, a problem does arise, and one which very often occurs in practice. The case most immediately connected with the general subject of this section is,

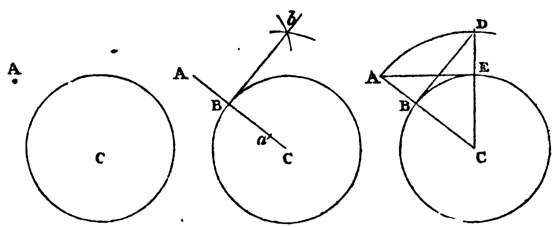
14. From a given point without a circle, to draw a line which shall touch the circle, but not cut it;—in other words, to draw a tangent to a circle from a point without the figure.

This is an application to the circle of a more general problem, namely, from one given point to draw a line which shall pass within a certain given perpendicular distance of another given point;—as, for instance, to lay down a straight line of road from one house that shall pass by another house at the distance of exactly four hundred yards. As the perpendicular is the shortest distance between a point and a line, the perpendicular from the house upon the line of road is the line which is to be four hundred yards long; but there is only one point in this line given, namely, the house, whose distance from the road the line is to represent. In the line of the road, also, there is only one point given, namely, the position of the house at which the line of road commences. Two points are necessary for determining the direction of a line; and therefore the positions of both of these are indeterminate; and all that the data inform us of respecting them is, that each must pass through a given point, and that the two must meet each other at right angles.

But the length of one of the lines is given, and one end of it is fixed at the house; therefore, if we imagine a circle to be drawn round the house at the distance of four hundred yards; and because the diameter, or, which is the same in direction, the radius, is at right angles to the tangent; the road must be laid down so as to touch this circle, but not cut it. This is, therefore, nothing but a particular case of the general problem of drawing a tangent *to* a circle.

Even this cannot be done directly; but we can draw a tangent *from* a circle, that is, from any point in the circumference; and when this is once done, we can accomplish the other.

In the first of the three figures following, C is any circle, and A any point from which a line is to be drawn to touch the circle.



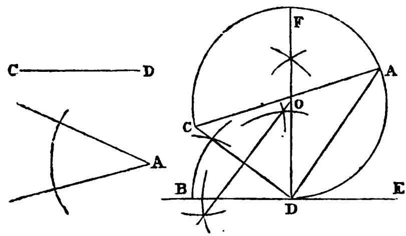
Repeat the circle as in the second figure. Join A C, cutting the circle in B; take B a = B A; from A and a as centres, describe arcs cutting each other in b; join B b; and B b is perpendicular to the radius at B, therefore it is a tangent to the circle at that point.

Again: repeat the circle as in the third figure, and draw the tangent at B as before. Then, from C as a centre, and at the distance C A, describe the arc A D, and continue it till it cut the tangent drawn from B in the point D. Join C D, cutting the circumference in the point E; draw A E; and A E is the tangent required,—that is, it is drawn from the point A, and touches the circle in E, but does not cut it.

The triangles C B D and C E A are equal in every respect, though they are symmetrical, and the one in great part overlays and conceals the other; and as C D and C A are equal, being radii of the same circle, and the angle at B opposite the one of them is a right angle, the angle at E opposite the other must be a right angle also. Therefore A E touches the circle at right angles to the radius, and thus it is the tangent which was required. There are other methods of performing this problem; but they require the application of principles which we have not yet investigated.

15. The problem of constructing upon a given line a segment which shall contain a given angle, and the converse, the cutting off of a segment from a given circle which shall contain a given angle, are both readily performed by means of the agreement of the angles made by the chord with the tangent, and those in the alternate segments into which the chord divides the circle, as mentioned in article 12 of this section.

In the case of constructing the segment which shall contain the angle, we have only to draw an indefinite line as a tangent; to make at any point in it an angle equal to that which the segment is to contain; and to produce the line forming that angle to exactly the length of the given chord. This being done, we have next to draw a perpendicular from the point where the chord meets the tangent, and the centre must be in this line. Lastly, we have to bisect the chord at right angles, and produce it till it meets the perpendicular to the tangent, and the point in which they meet will be the centre, from which the segment may be described on the alternate side of the chord.



Upon C D, to construct a segment that shall contain an angle equal to that of A.

Draw B E as the tangent, and at the point D in it make the angle B D C equal the angle at A, and D C equal the given chord C D. From D draw D F perpendicular to B E; bisect C D at right angles, and produce the bisecting line till it meets D F in O, and O is the centre. From O with radius O D, describe D A C; and and D A C is the segment required; for it is the alternate segment to the angle B D C, which is equal to the given angle, and it is constructed on D C, which is equal to the given line C D.

The points which we have mentioned in this section comprise almost all the elementary principles of the simple intersection of lines and circles, which are absolutely necessary. There are, however, a few more that are very nearly self-evident in their nature, which it may be proper to bear in mind.

1. All circles which have the same radius, or equal radii, are equal in diameter, in circumference, in area or surface, and in every respect; and if two circles can be shown to be equal to each other in any one of these respects, it may be inferred, without any farther proof, that they are equal in each and in all of the others.

2. In equal circles, corresponding parts are also equal to each other, whether they be parts of the area, the circumference, the diameter, or of any two lines placed similarly in the circles. We must bear in mind, that any portion cut off from a circle by a straight line or chord which meets the circumference both ways, is a segment of the circle; and that any part cut off by two radii which meet at the centre of the circle, is a sector, which means "cut into," in like manner as segment means "cut off." This being borne in mind, it will follow, from what has been stated above, that

3. In equal circles, equal straight lines cut off equal segments, both in respect to the true segments which contain the centres, and of those which do not; and that whether the equal segments be cut from corresponding parts of the circles or not.

4. In equal circles, equal sections are formed by radii making equal angles at the centres, or, which is the same thing, stand upon equal arcs of the circumference; for, as the angle and the arc are mutually and reciprocally the measures of each other, it follows, by necessary consequence, that whatever is in the proportion of the arc, and regulated by it, must also be proportional to, and regulated by, the other. Hence,

5. If two lines cross or intersect each other in the centre of a circle, and are both produced till they meet the circumference both ways, the vertically opposite sections which they form must be equal in every respect. They are equal in their angles, because the vertically opposite angles which are formed by two lines which cross each other are always equal; they are equal in the arcs of the circle, because the angles which are the measures of those arcs are equal; and they are equal in their straight sides, because those sides are all radii of the same circle, and therefore equal by the fundamental property of that figure.

The argument for equality, in all these cases, is the "sufficient reason," namely, that there is sufficient reason for asserting the equality of the quantities in each case in which they are said to be equal, and no reason whatever of denying, or even for doubting, their equality. In every part of each of these five cases which have been enumerated, the argument founded on the sufficient reason is complete. We are in possession of all the circumstances which determine the value of each of the quantities; and those circumstances are exactly the same, in number, in order, and in extent, in every two quantities which are alleged to be equal. Now, this is perhaps not quite so simple a proof of equality as that which we obtain when we apply one magnitude to another, and find them to coincide exactly. It is, however, a far more general, and therefore a far more useful ground of equality than the other; for though the notion of equality which we obtain by supra-position, or the placing of one magnitude upon another, and seeing the perfect coincidence of all their boundaries, answers very well in the mere elements of geometry, yet it does not carry us very far even there; and we are quite unable to apply it to almost any one practical case. Thus its apparent simplicity is of that simple kind of which little or nothing can be made.

It does not apply to any description of magnitudes, except straight lines, angles, and surfaces; neither does it apply to equal surfaces, bounded by equal lines, and having equal angles; unless these are turned the same way. Besides, the idea of motion is always involved in this proof of equality; for there is no other way in which we can imagine one line, one angle, or one surface to be applied to another, but by lifting the one of them, and actually making the application in a mechanical manner, very much in the same way as the length of a board, or any other substance, is ascertained, by applying to it a foot rule, or any other instrument for the measuring of length.

Even in geometry, it is only a very little way that this doctrine of equality will carry us; for when we come to the equality of lines, at the extremities of which there are unequal angles, we cannot suppose the one figure to be placed on the other so as that we can derive, from the placing of it, any useful conclusion whatever. This holds in the case of triangles, which are the most simple and the most determinate of all straight-lined figures. It is true, that if the three sides of one triangle are equal to the three sides of another, the two triangles must always be equal to each other in every respect; except that of lying in different directions, which has nothing to do with the question of magnitude, or with that of shape. But the converse of this does not hold; for though triangles which have the sides different from each other, are necessarily different in shape, and in their angles, and thus cannot be applied to each other by supra-position; yet still they may be exactly equal in area, or in the surface which they contain within their bounding sides. This is of itself sufficient to show us that the principle of coinciding, or occupying the same space, in the ordinary signification of the words, cannot be made the general ground of equality, even in surfaces; but that it is strictly confined to straight lines, rectilineal angles, or like parts of the same circle, or of equal circles.

Nor is it difficult to see why this must be the case; because our notion of a surface, considered as a magnitude, is a compound one, involving the idea of two factors as elements, and the multiplication of those factors, in order to obtain a product, which product is the proper expression for the value or magnitude of the surface. Now we have seen already that the same product may be obtained from an endless number of factors, provided that the one is increased, and the other diminished, in due proportion. Thus, if any surface or area whatever is expressed by the product of two definite factors, a and b, that is, by $a\,b$; and if we state the proportion;

$$m : a = b : n,$$

then, to every possible value of m, however great or however small, there will be a corresponding value of n, which will make the product $m\,n =$ the product $a\,b$; and if we have the value of m given, the corresponding value of n can always be found by a simple application of the rule of three; for in all cases $n = \frac{a\,b}{m}$.

If supra-position as a test of equality will not apply to surfaces, of which our notion as definite magnitudes is always the result of a single multiplication only, much less will it apply to solids, of which our notion is that of the product of two multiplications of three factors, standing to each other in the relations of length, breadth, and thickness.

In the case of ratios it is still less applicable; because the ratios are not in themselves quantities at all: they are merely the results of the comparison of quantities, and therefore they do not occupy space in any sense of the word. Geometry would, however, be very imperfect and very useless, if it did not embrace the doctrine of ratios; and it really does appear that the tying down of the primary notion of equality in the student to this very limited fact of coincidence, is the chief reason why so much difficulty is always felt with the fifth book of Euclid's Elements, though that book is in reality the simplest of the whole.

When we come to the general expression of quantities algebraically, which is by far the most valuable portion of the mathematical sciences; or to the arithmetical expression of them, which is the only means whereby the principles can be reduced to practice; the notion of occupying the same space can have no application or meaning whatever; because neither the algebraical expressions, nor the figures of arithmetic, have the slightest reference to the occupation of any space, large or small; and therefore we have to judge of their equality or inequality upon very different principles. The one guiding principle in these is, that equal data equally dealt with invariably lead to the same results; and this ground of equality is perfectly satisfactory, and possesses the advantage of being applicable to all cases, whether they be of a mathematical nature or not. Thus, it brings geometry, as well as every other branch of mathematics, to the very same standard upon which we found our judgments in all the conduct of life; and consequently, instead of making geometry stand apart, as if it were unconnected with our ordinary modes of thinking and acting, it brings it home to the mind as part of that general education which we derive from observation and experience, without in the least affecting that rigid accuracy which is the valuable part of mathematical study. It is impossible to extend this primary geometrical test of equality beyond the subjects of lines and angles, or to bring it in any way to bear upon our common modes of judgment; and surely, therefore, the wise plan is, to endeavour to bring these modes of judgment to bear upon geometry as well as other matters, in order that our modes of thinking may be the same upon every subject. By this means we are left perfectly untrammelled, to direct our attention to the peculiarities of the subject itself; and if there were nothing more than this to be gained, it would be well worthy of all the labour that it costs.

We have alluded to this subject more than once in the previous part of this volume; but as we are now to proceed more directly to subjects which are geometrical, we have felt it necessary to dwell upon it at some length, in order that the reader may have it fresh in his memory, and thus be prepared for availing himself of any and every advantage which it may afford. We purposely treated of the doctrines of ratios, and of powers, and roots, at least in their most elementary forms, at an earlier stage than is usually done in books of geometry. We did so, in order that we might carry our doctrine of proportion along with us as an element; and we feel convinced, that any one who chooses to look into the fifth book of Euclid's Elements, where he will find that, with the exception of one of the axioms, and a single proposition, which is very nearly self-evident, the four preceding books of the elements are not even once alluded to in the fifth one, which, in fact, stands alone, and has really nothing to do with the properties of figures; as the ideas in the axiom, and the proposition to which allusion is made, are both strictly arithmetical in the particular case, and algebraical as taken generally.

Having given these explanations—which, however, are intended more for the guidance of the reader, than as an apology for departing from the established order of succession in the elements of geometry—we shall proceed to another section.

SECTION XVI.

COMPARISONS AND RELATIONS OF PLANE FIGURES, THEIR BOUNDING LINES, THE ANGLES MADE BY THOSE LINES, LINES INTERSECTING THEM, AND THEIR SURFACES OR AREAS.

In order that we may proceed with the requisite ease and expedition, in dealing with the subjects enumerated in the title of this section, it will be necessary that we carry along with us some preliminary notions, in addition to what have been treated of in the former sections. This is the more necessary, on account of the general view which we wish to take of the doctrine of equality, which doctrine also involves in it the opposite doctrine of inequality, in as much as where the one ends, the other begins as a matter of course.

With regard to straight lines, to circles, and to angles as determined by circular measures, we believe that we can hardly make the matter plainer than we have already attempted to do; or indeed than it must appear to every one who understands the definition of a line, a circle, or an angle, when either of them is presented singly to the mind.

When, however, we come to compare figures, or the sides or angles of figures, the quantities which we have to compare are results, and not simple data given singly; and consequently, in them the comparison of the results may involve, and very often does involve, the comparison of the means by which these results are arrived at. Indeed, it is this compound view of the matter in those cases, which gives them the greater part of their value; and therefore, if we do not familiarize ourselves with the result, so as to be able to analyze all the steps, and compare them, each with each, in the two parts of our general comparison, we never can be certain or satisfied that we are right; and thus, while we have the semblance of mathematical demonstration, we have nothing in reality but simple belief. This is another formidable obstacle in the way of the student of elementary mathematics; and it is one, the removal of which is of far more importance than any apparent progress in detached theorems and problems, which could possibly be made by one before whom this obstacle were always presenting itself.

GENERAL MAXIM.

The same data which are sufficient for enabling us to construct any figure, are also sufficient for establishing the perfect equality in every respect of two figures of the same species; that is to say, if the data are all exactly the same in the case of the one figure as in that of the other.

This is a simple and general principle, not affected by any contingency; and, therefore, like all such principles, the converse of it is true; that is to say, the same data which suffice for establishing the perfect equality in every respect of two figures, are quite sufficient for enabling us to construct those figures.

The truth of this maxim, viewed both directly and conversely, is so clear and simple, that it cannot be made more so by any attempted demonstration; and yet it is one which, though seldom stated and generally overlooked, is very useful in shortening and simplifying many reasonings in mathematical science. When we consider it, we can readily see that it is of very general application, not to mathematical subjects only, but to all subjects in which any thing has to be done. If for instance, one states a proposition in writing, makes a drawing, or performs any other operation, in which a result is arrived at by the use of means; then this result is nothing more than a copy of the mental perception which was previously had of the use of the means, and the result that, as a matter of certainty, followed the use of them; and if this is done with sufficient knowledge and judgment, the person doing it can repeat it again and again with a feeling of perfect certainty that the result must be perfectly equal in all cases.

Though this maxim is perfectly general, yet we must understand it within the proper limits before we can depend upon it for that absolute certainty which belongs to the mathematical sciences, and to them only; though the nearer that we can approximate this certainty in other matters which are of a mixed nature, the better. This, by the way, is the grand practical use of mathematics, and of incalculably more value to mankind than all the technical applications, which are absolutely necessary upon comparatively few occasions in the common business of life.

Let us apply this general maxim to the simplest of all cases, namely, the ascertaining of what data are necessary for constructing, and for ascertaining the perfect equality, in every respect, of

TRIANGLES.

1. If the three sides are given, we have already shown how the triangle may be constructed, with the limitation of this single condition—that any two of the three sides must be greater than the third one. We have to place one of the sides on a straight line, and on the extremities of this side, as centres, describe circles, having the radii equal to the other two sides, one at the one extremity and the other at the other; and if the intersection of those circles is joined by two lines drawn to the extremities which were made the centres of the circles, a trian-

gle is constructed, the three sides of which are equal to the sides which were given.

It is of no consequence in what order the three given sides are taken, because the triangle must have exactly the same size and shape, in all cases of the same three sides; but it may have several positions, which do not, however, in the least affect its form or its value.

If the same side is placed on the line, in order that its extremities may be the centres of the circles, there are four positions of the triangle in the case of its being *scalene;* that is, of having all its sides unequal. That is to say, the triangle may be constructed upon either side of the line, and the longer or the shorter of the two remaining sides may be placed at one extremity, that is, made the radius of the circle there. Thus the triangle may be turned side for side, and also end for end, in both cases, which gives four positions; but as the sides are all the same in each case, and subtend arcs of equal circles, it follows that the angles opposite the equal sides are equal in each of the four positions.

But any of the three sides may be placed on the line, in order to give the centres of the circles whose radii are made respectively equal to the other two sides; and this again gives three varieties of position in each of the former four, or twelve in all, though there is nothing in either which can produce the slightest difference in the size and shape of the whole triangle, or of any of the six parts—the three sides and three angles of which it is made up.

In all these twelve positions of the triangle, we have assumed that the position of the original line to which the side first used is applied to remain the same; but this line may have any direction whatever, without in the least affecting the value of the triangle, or of any part of it; and, therefore, the same triangle may have an indefinite number of positions. This, without any reference to the fact that position is not a datum in the construction of the triangle, would suffice to show that position has nothing whatever to do with form or the value; but that the triangle, of which the three sides are known, must be the same, in what part of the world soever it is constructed, and whether it is ever constructed or not, provided that the lengths of the sides are expressed in the same measure and properly understood. Thus, if the three sides of a triangle are respectively, 5 inches, 4 inches, and 3 inches, which have the condition necessary for forming a triangle—as any two of them are together greater than the third, then this triangle has a definite shape and size from which it cannot deviate; and any one who remembers the numbers, and is possessed of a rule or scale divided into inches, and a pair of compasses for describing circles, can construct this triangle whenever he pleases; and if he should describe one in London, and another at any distant place—say Calcutta in the East Indies—he would have no more doubt of the equality of those triangles in every respect, than if they were cut out in two pieces of flat paper, so that the one could be applied to the other, and be seen to coincide with it, or fill the same space in the whole and in every part.

These numbers, 5, 4, and 3, expressing the three sides of a triangle in the same measure, are worthy of being remembered; for, as we shall see afterwards, the angle opposite to the side 5 must be a right angle; and therefore we can always get a right angle by constructing a triangle whose sides are 5, 4, and 3; and as they are the simplest lengths of sides which have this property, they are worth bearing in mind.

2. If two sides, and the angle included between them, are given, there are sufficient data for constructing the triangle;

and if two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other —it being understood that it is not the sum of the sides which is equal, but that each of the two sides of the one triangle, taken singly, is equal to one side of the other—then the triangles are every way equal.

This does not require a formal demonstration, or the application of the one triangle to the other, any more than the former case; for, if the lengths of the sides and the measure of the angle between them are known, the triangle can be constructed at any time or in any place; and if two triangles, with equal including sides, and an equal included angle, are constructed—or even imagined to be constructed—how far soever they may be asunder, we can no more doubt their equality than we can doubt that an inch is an inch, or any angle itself and not another.

The given sides determine their own lengths, and the given angle determines their position with regard to each other. By this means the position of those extremities of them which are most distant from the angle are also determined, and these determine the length of the third side which joins them. They do this not only in respect of the length of this third side, but in respect of its position with regard to each of the two given ones, at the points where it meets them; and this, of course, determines the remaining angles of the triangles; and if there are two such triangles, it necessarily follows that the third sides of both are equal, and also that the angles opposite the equal sides are equal.

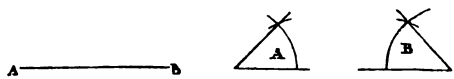
3. If one side and two angles of a triangle are given, there are data sufficient for constructing the triangle; and if two triangles have a side and two angles of the one equal to a side and two angles of the other, they are equal in every respect.

If two angles are given, then the third is also given, or, which is the same thing, there are data sufficient for finding it; for the three angles are equal to two right angles, or the angular space on one side of a straight line at a point is the same. Therefore, to find the third angle, we have only to draw a straight line, and take any point of it; then, at this point we make an angle equal to one of the given angles; and afterwards apply to this angle, at the same point, another angle equal to the second given angle; and the angular space between this last angle and the opposite end of the straight line, which is the supplement of the sum of the two given angles, must be the third angle of the triangle. When this is done, the given side and the three angles are quite sufficient for enabling us to construct the triangle, whether the two given angles are the ones adjacent to the given side—that is, the one at the one extremity of it, and the other at the other—or whether one of the given angles is opposite to the given side.

In order to construct the triangle, it must be stated in what order the given angles are to be arranged with regard to the given side. If they are both adjacent to it, the form and magnitude of the triangle are determined, but not the position, because that would be reversed, if the given angles were unequal and made to change places. Also, if only one of the given angles is to be adjacent to the given side, we must know to which extremity of it that angle is to be adjacent, otherwise we may reverse the position of the triangle. In every case, howver, the magnitude of the triangle is determinate; and in comparing two triangles, which have the three angles and a side in the one equal, each to each, to the three angles and a side of the other, the sides opposite equal angles, and also the angles opposite equal sides, are equal to each other, and consequently the triangles are equal in every respect.

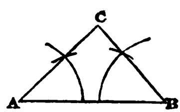
We shall illustrate this by a diagram, as it is a little, though only a very little, more complicated than the preceding ones. The only limitation required in this problem is, that the two given angles shall be together less than two right angles; for, if they are not, then they cannot be two angles of a triangle, as all the three are just equal to two right angles.

This being understood, let the line A B be the given side; and the angles at A and B, the given angles; it is required to construct the triangle.



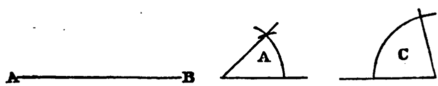
First, let the two given angles be adjacent to the given side, A at the extremity A, and B at the extremity B; and in this case it is not necessary to find the third angle, as it will be determined by the construction.

Make A B equal the given side; with any radius describe arcs on A and B, the given angles; and with the same radius describe arcs on the points A and B of the repeated line; cut off portions of these arcs, equal respectively to the arcs intercepted by the given angles A and B; and through A and B, the extremities of A B, and the points of section, draw lines till they meet in C; and C is the third angle, A C and B C the remaining sides, and the triangle is constructed.

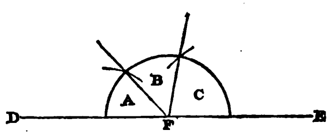


The angles at A and B in this triangle are equal to the given angles at A and B, for by construction they stand upon equal arcs of equal circles. Also, the lines A C and B C must meet, because the interior angles which they make with A B, toward the side C, are together less than two right angles. They must meet in some one point C, because two straight lines can meet in one point only; and as their positions are both determined by the given angles at A and B, and the point C must be in them both, they determine the place of the point C, and that point determines the length both of A C and B C; consequently the whole triangle is determined.

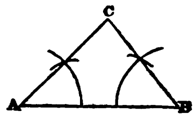
Secondly, Let one of the given angles—as, for instance, the angle C—be opposite the given side A B, and the other adjacent to it, as the angle A, at the extremity A of the given side; A B is the given side, and A and C the given angles.



The first thing to be done here, is to find the remaining angle at B; and after it is found, the triangle is constructed exactly as in the former case. To find the angle B,

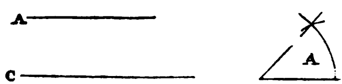


draw any line D E, and in it take any point F, but not at either extremity. With any radius, describe arcs A or C on the given angles A and C; and with the same radius describe a semicircle on the line D E from F as a centre. Make the arc A of the semicircle equal to the arc of the given angle A, and the arc C of the same semicircle equal to that of the given angle C. Draw lines from F to mark the terminations of those arcs; and the remaining arc B represents the angle B of the triangle; for it is the supplement of the given angles A and C to two right angles. Apply the arc A on the same radius to the extremity A of the given side, and the arc B to the extremity B of the same; through A and B, and the extremities of the respective arcs, draw lines; and those lines will meet in C, and form the triangle required.



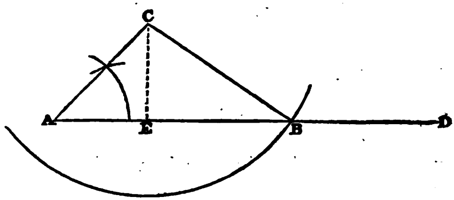
As the only difference between this case and the former is the finding of the angle B previous to the construction of the triangle, it requires no further explanation.

4. If two sides and an angle opposite one of them are given, the triangle may, in some cases, be constructed with the same certainty as by means of the data in any of the three preceding articles; but there are other cases in which the triangle obtained from these data is ambiguous, or admits of two different forms and magnitudes; and there are also cases in which the construction is impossible, so that there can be no triangle. The same ambiguity and the same impossibility will, of course, occur when we attempt to prove the equality of two triangles from two sides, and an angle opposite one of them; and therefore this mode of proof can be applied only in certain cases; and it is necessary to know what those cases are; for in them the proof is as clear and as complete as it is by any other data. We shall best explain this by taking data, and constructing the figures as we proceed.



Let A and C be the two given sides, and A the given angle. And first, let the angle A be adjacent to the less side A, and consequently opposite to the greater side C; and this is the determinate case in which the triangle can always be constructed, and has only one form and magnitude.

Draw any line A B, and produce it to such a length as may be judged convenient.



Then, at the point A in the line A D, draw A C, making with A D an angle equal to the given angle at A; and make A C equal in length to the given side A. From C as a centre, and with a radius C B equal to the given side C, describe an arc cutting A D in the point B; join C B, and A B C is the triangle.

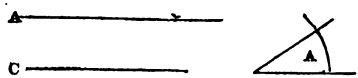
This may require some further explanation. The arc described with the greater of the given sides C as a radius, must cut the line A D if produced far enough; but it can do so only in one point between A and D; so that the point B cannot have two situations, and consequently the side A B, or the triangle A B C, cannot have two values.

To prove this: from C draw the dotted line C E perpendicular or at right angles to A B. C E is less than C B; because the angle C E B opposite C B is a right angle; and consequently the angle C B E must be less than a right angle, and the less side is opposite the less angle; consequently C E is less than C B; and the point E must fall within the circle of which C is the centre, and C B the radius. But the point E is in the line A D, and therefore the circle must cut or cross that line, before it can be on the opposite side of E from C.

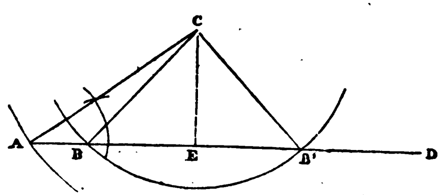
Again: the point A must fall within the circle, because A C is by hypothesis, that is as the data were taken, less than C B the radius. Therefore the circle cannot cut A D a second time between A and D.

The same figure will show us the state of the data which leads to the impossible case. If the given side C had been the less of the two, and less than the perpendicular C E let fall from C upon A D, a circle with this radius could not have cut the line A D; and consequently there could have been no point B, and no triangle.

Secondly, Let us examine the case in which the given angle is adjacent to the greater of the given sides, and opposite to the less. This is the ambiguous case in all states in which the opposite side is less than the adjacent one, but not less than the perpendicular; for when it is less than that, it becomes the impossible case.



Let A and C be the two given sides, and A the angle adjacent to the side A, but let A be greater than C.



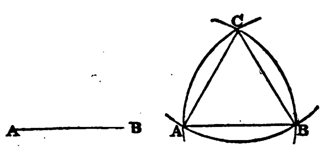
Draw A D as before, make the angle at A equal to the given angle, and A C equal to A the greater of the given sides; and C is one point of the triangle. From C let fall the dotted perpendicular C E; and if C, the less of the given sides, is less than C E, there can be no triangle. Let it be greater than C E, but less than C A; then, from C as a centre, with a radius equal to C, describe an arc; and this arc must cut A D, and will cut it in two points, B and B′, on opposite sides of the point E, and equally distant from it. So that the triangle is either A B C, of which the side situated on A D is the smaller portion A B; or it is A B′ C, of which the side situated in the line A D is the greater portion, A B′, made up of the three parts, A B, B E, and E B′; and the difference of those two triangles is the triangle C B B′, of which the sides C B and C B′ are equal to each other, being radii of the same circle.

It is also evident that the arc described on C as a centre must cut the line A D between A and E; for if an arc is drawn about C, with the radius C A, the point B must be within the circle of which that arc is a part, because C B is less than C A the radius.

These four articles embrace all the data, furnished simply by the parts of a triangle, and without reference to any other figure, which apply generally to the construction of all triangles, whatever may be their form; and by which the perfect equality of any two triangles which possess them may be established. We have seen that all of them require some conditions or limitations, which are necessary before there can be a triangle: namely, that no one side shall be greater than the other two, and no two angles equal to or greater than two right angles. We have farther seen, that in the last of the four, it is required that the given angle shall be opposite to the greater of the two given sides; otherwise the triangle is either ambiguous, having two forms and values, or the data are in-correct, and it is impossible. There is, indeed, one single case of the less side opposite the given angle in which the triangle is possible and not ambiguous; and that is, when the side opposite the given angle is exactly equal to the perpendicular C E, and the arc touches the line A D, but does not cut it. In this case, the angle opposite the greater given side is a right angle; and therefore the three angles of the triangle are virtually, if not expressly, given; and therefore this is a peculiar case, and properly belongs to that class of data in which the three angles are given.

There are some particular triangles, however, which, from their simplicity or regularity, can be constructed with fewer data; and it may not be amiss to mention at least one or two of them. The first and simplest is an equilateral triangle, the only datum necessary for the construction of which is the length of one of the sides. For, if one side is given, all the sides are given; and all the angles are also given, for they are necessarily equal to each other, in consequence of the equality of the sides; and each of them is the third part of two right angles, or the sixth part of four. There are some not unimportant conclusions to be drawn from this, to which we shall very briefly advert, after constructing the triangle.

Let A B be any straight line, it is required to construct an equilateral triangle, having each of its sides equal to A B.

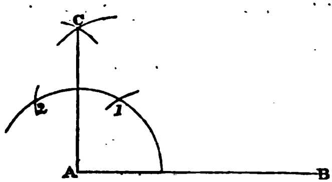


Take another line equal to A B, and from A and B as centres, with a radius equal to A B, describe arcs cutting each other in the point C; and join A C and B C; and A B C is the equilateral triangle required.

If each of the three angular points is made the centre, and an arc described upon each as a centre, meeting the other two, each of those arcs is the measure of the third part of two right angles; that is, of 60° of an equal circle, of which the side of the triangle is the radius; but as viewed from each angle, the side opposite that angle is also the chord of 60°. Therefore, the chord of 60° is equal to the radius of a circle; and if the radius of a circle is applied six times to the circumference, it will subtend the whole circumference in six equal parts of 60° degrees each; and if we join the adjacent points of division all round, we shall have inscribed a regular hexagon, or figure of six equal sides, within the circle.

The circumference of a circle can always be divided into two equal parts by drawing a diameter till it meets that circumference both ways; and the radius, applied as above stated, divides the circumference into six equal parts. Thus we have an easy means of dividing a circle into any number of parts, which number is either a multiple of six by two, or a quotient arising from the division of six by two. These matters are so simple, however, that any one may readily understand how they are to be done, without further explanation than has been now given.

This property of the chord of 60° being equal to the radius, affords us a very convenient method of erecting a perpendicular at any point of a given line, whether that point is or is not the extremity of the line. Thus, let A B be any line, and let it be required to erect at the extremity A of the line, another line perpendicular to A B.



On A as a centre, with any radius not greater than A B—but the greater that it is the operation will be the more accurate,—describe an arc, equal to at least a third of a circumference, from A B toward that side where the perpendicular is required. Apply this radius on the arc from the line A B to the point 1, and again from the point 1 to the point 2. Then from 1 and 2 as centres, and still with the same radius describe arcs cutting each other in the point C; join A C, and A C is perpendicular to A B at the point A. For the arc from the line A B to 1 is 60°, and so is the arc from 1 to 2. But the line C B bisects the arc 1, 2; and therefore the portion of the arc intercepted between A B and A C is 60° and the half of 60°, or 90°, which is the measure of a right angle; therefore B A C is a right angle, and A C is perpendicular to A B. This is one of the most convenient methods of obtaining a right angle in practice.

In the case of an isosceles triangle, or triangle having two equal sides, the only data required for the construction or for establishing the equality of two triangles, in the case of the data being the same, are the unequal side and one of the equal ones; for if one of the equal sides is given, the other is given; and thus the problem resolves itself into the constructing of a triangle of which the three sides are given, and thus we have only to apply the equal side as a radius at each end of the unequal one, and describe arcs intersecting each other, in that direction in which we wish the triangle to be situated; and when both extremities are joined to the intersection of the arcs, the triangle is constructed. The only condition or limitation required in this case, is that the equal side shall be greater than half the unequal one; because if it were not, the arcs would not intersect each other.

The next consideration with regard to triangles, is the equality of their surfaces when the sides and angles are different; but before we proceed to this, it will be desirable to consider some of the more elementary properties of

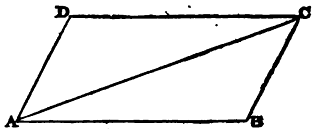
PARALLELOGRAMS, RECTANGLES, AND SQUARES,

and the relations which they have to triangles.

A Parallelogram is a four-sided figure, of which the opposite sides are parallel; and a line joining the opposite angles either way, is called the diameter of the parallelogram. This figure is usually named by repeating four letters placed at the angles, and taken in regular order round it; or more briefly by two letters situated at opposite angles. In a parallelogram, every side is adjacent to two other sides, and opposite to the remaining one, that is, the one to which it is parallel; and every angle is adjacent to two angles, and opposite to the remaining one; also, there are two sides opposite to every angle. Therefore, though the sides of a parallelogram are given, the figure is not given—that is, there is not data enough for the construction of it, unless at least one of the angles, or one of the diameters, is given. It will, therefore, be necessary to attend to some of the properties of the figure, before we proceed to the comparison of one parallelogram with another, or of parallelograms with triangles.

1. The opposite sides and angles of any parallelogram are equal to each other, and the diameter joining the opposite angles either way divides it into two equal parts.

Let A B C D be any parallelogram; the sides A B and D C are equal to each other, and so are the sides A D and B C. Also,



the opposite angles at A and C, and those at B and D, are equal to each other. And if a diameter is drawn from A to C, or from B to D, that diameter will divide the surface of the parallelogram into two equal parts; namely, into two triangles, having all their sides and angles equal, only placed in reversed positions upon the diameter.

Because D C and A B are parallel, the angles at D and A are together equal to two right angles, and because A D and B C are parallel, the angles at A and B are also together equal to two right angles. Leave out the angle A, which is common to those equals, and the remaining angle D is equal to the remaining angle B. But, because of the parallels, the angles B and C are together equal to two right angles, or to the two angles at D and A. From these equals, take away D and B which are also equal, and the remainders, which are A and C, must be equal.

But again: the alternate angles D C A and C A B on the parallels D C, A B, are equal, and so are the alternate angles D A C, A C B, on the parallels D A, C B. Therefore, the two triangles A B C and C A D, have the side A C common, and all their angles equal; therefore, the sides opposite the equal angles are equal, that is, A B in the one is equal to C D in the other, and B C in the one equal to D A in the other. Consequently these triangles are equal to each other in every respect; and as the two together make up the whole parallelogram, each one singly must be equal to the half of it. Therefore, the opposite sides and angles are equal, and the diameter bisects the parallelogram.

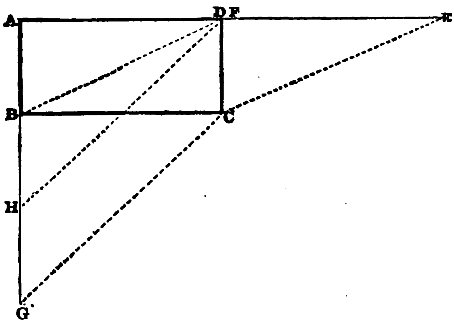
The converse of this, that straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel, follows, as a matter of course, from this equality of the opposite sides of a parallelogram.

2. If a parallelogram has one right angle, all its angles are right angles.

This follows immediately from what was shown in article 1. The opposite angles are equal, so that if one is a right angle the one opposite to it must also be a right angle; but the four angles are altogether equal to four right angles, and these two opposite ones are equal to two, so that there remain other two right angles for the remaining two opposite angles of the figure; but they, too, must be equal to each other, or each of them must be a right angle. Consequently, if one angle of a parallelogram is a right angle, each of the four must be a right angle.

3. A parallelogram, which has its angles right angles, is called a *rectangular* parallelogram, or simply a RECTANGLE; and a figure of this kind is said to be contained by the two sides which are about or include any of its angles. This last expression has reference to the area or surface of the rectangle; and the one of the containing sides is called the base, and the other the altitude or height, and the one may always be considered as representing length, and the other as representing breadth. Thus, when we speak of the rectangle A B C D as a surface, we call it the rectangle A B, B C, which means that the expression for the surface is the product of A B multiplied by B C, that is, A B $\times$ B C, both being understood to express the same kind of measure as lines, and the surface as many squares of the lineal measure, as the product of the numbers. This is the original connexion between Geometry and Arithmetic; and the princi-

ples of it have been explained in a former part of this volume, so that we need not dwell upon it at present, nor adduce any example in illustration, farther than the following simple figure:



A B C D is any rectangle whatever, of which B C is the length or base, and A B the breadth or altitude. The value, that is, the area of the rectangle, must remain the same, while the length and breadth remain the same, or, which is the same thing, equal; for the area is the product of those dimensions or factors, unaffected by any thing except their individual lengths and the fact of their standing to each other in the relation of length and of breadth; and while they remain equal and preserve this relation, there is no circumstance which can in any way alter the area or value. Thus, for instance, if A D is produced towards E, still parallel to B C, but to any distance whatever, and C E joined as by the dotted line, and B F drawn through B parallel to C E, then the oblique parallelogram F B C E must also have the same area or surface as the rectangle A B C D. Also, if A B is produced indefinitely, and two parallels drawn from D and C, in any direction, but both in the same, till they meet A B produced in the points H and G, the parallelogram H D C G thus formed must be equal in surface to the rectangle A B C D, in what place or in what position soever it happens to be situated; for we have already shown that place and position, are not elements which can affect the value of any magnitude or quantity.

This, which is a very useful property, is usually cited in the words, "Parallelograms upon the same or equal bases, and between the same parallels, are equal to each other." But there is some objection to the words "between the same parallels," inasmuch as, to a beginner, they tie down the equal parallelograms to one locality; and, therefore, the words "equal bases and altitudes," or "equal lengths and breadths," are preferable, inasmuch as they are not tied down to any particular locality, but can be understood and applied whether the parallelograms are between the same parallels or not.

From mere inspection of the equal parallelograms in the above figure, it will be seen that while the area or surface remains of the same value, the lengths of the sides may undergo changes. Thus, for instance, in the oblique parallelogram B C E F, the parallel sides which are dotted, namely B F and C E, are each greater than B A or C D; for they are opposite right angles at A and D, while B A and C D are opposite angles at F and E in the same triangles, each of which is less than a right angle. But A B might have been produced any length to G, so that the continuation is still parallel to C D. C G might be joined and D H drawn from D parallel to C G, and meeting A G in H; and the parallelogram G C D H must still be equal in area to the right angle A B C D; for the two dimensions, of which its area or surface is the product, remain unchanged, both in their lengths and in their relation of base and altitude, or length and breadth; while the sides which bound the oblique parallelogram, and make angles less than right angles with the parallels, are always the greater the less the angles that they make. Hence we have this general conclusion, that if in any figure whatever we can get two dimensions which stand to each other in the relation of length and breadth, and correctly represent the length and breadth of that figure as a rectangle, then the product of those two dimensions will always be a correct expression for the area or surface of the figure, whatever may be the shape of that figure, provided only that it is bounded by straight lines.

It follows from this, that any straight lined figure whatever may be reduced to a rectangle of exactly the same value, and its surface as a whole expressed in square measures, that is, measures which are the squares of some known measures of length. This is the principle which applies to the measuring of land, and of all other surfaces whatever, whatever may be the form and dimensions, if the surface is a plane, or regarded as such.

4. Upon looking back at the diagram in article 1, page 428, it will be understood that while A B is the length or base of the parallelogram, a perpendicular let fall from D upon A B, or from C upon A B produced, must express the altitude or breadth of the parallogram A B C D; and if we call this perpendicular p, then the area or surface of the parallelogram will be expressed by the product A B $\times$ p.

Now the triangle A C B, which is on the same base with the parallelogram and has the same altitude, was shown to be equal in surface to half the parallelogram; and if the base and altitude had been equal, the surface would have been the same, as the mere situation, by not affecting either the value or the relation of either of the factors, cannot possibly affect the value of the product of those factors. Hence we have the general conclusion: if a triangle and parallelogram have equal bases and altitudes, the area of the triangle is half that of the parallelogram.

From this we can immediately derive a method of finding the area either of a parallelogram or triangle, when the base and altitude are known. It is: for the parallelogram, multiply the base and altitude; and for the triangle, multiply the same, and take half the product—or, as it is the same to divide either factor before multiplying as to divide the product after, we may multiply the base by half the altitude, or the altitude by half the base, and the product in either case will be the area of the triangle.

4. But, in the case of a product, we multiply the product if we multiply either factor, and divide the product if we divide either factor. Therefore, triangles or parallelograms which have equal bases, are to each other in the ratio of their altitudes; and triangles or parallelograms which have equal altitudes, are to each other in the ratio of their bases. Therefore, again—

Triangles or parallelograms which have different bases, are to each other in a ratio compounded of the ratios of their bases and altitudes. Thus, for instance, if the base of the triangle or parallelogram A is 12, and its altitude 6; and the base of the triangular parallelogram B is 9, and its altitude 4—the numbers meaning equal measures in all those cases; then—

$$A : B = 12 \times 6 : 9 \times 4 = 72 : 36;$$

that is, the area of the triangle or parallelogram A is double that of the triangle or parallelogram B; and any others whose dimensions are known and expressed in the same measure, can be compared in the same manner; and those products which are compared are, after all, nothing more than the simple expressions for the areas of the figures as already explained.

5. If two triangles or two parallelograms are equal in area, but have their bases and altitudes different, then the bases and altitudes are inversely or reciprocally proportioned. Let a be the base and b the altitude of the one; and c the base and d the altitude of the other; then, if the areas are equal, that is, if

$$a\,b = c\,d, \text{ then}$$

$$a : c = d : b,$$

for multiplying the extremes and means, we have $a\,b = c\,d$, as before.

6. A rectangle which has two sides containing one of its angles equal, has all its four sides equal, and is called a SQUARE; and the expression for the area of a square is the product of the side multiplied by itself; or if the side be called a, then the area of the square is expressed by a^2.

If a square and rectangle have equal areas, the side of the square is a mean proportional between the base and altitude, or which is the same, between the length and breadth of the rectangle. For since they are both rectangles, and equal in area, the sides or factors whose products express their areas, are reciprocally proportional; that is, if b is the length and c the breadth of the rectangle, and a the side of the square, then

$$b : a = a : c;$$

and, multiplying the extremes and means, we have $b\,c = a^2$, or the areas are equal, which was the proposition.

Before proceeding further with the general principles, we shall endeavour to show how the last mentioned figures may be constructed. To construct a square on a given line. At one extremity of the line draw a perpendicular toward that side of the line upon which the square is to be situated, and make this perpendicular equal to the given line. Then from the two extremities of those lines as centres, and with a radius equal to either of them (for they are the same), describe arcs intersecting each other; the point of section of those arcs will be the fourth angle of the square; and if lines are drawn from this to the second and third angles, the square will be constructed.

To construct a rectangle under two given straight lines. Take in any straight line a portion equal to one of them, and at either extremity erect a perpendicular equal to the other; then, on the opposite extremity of each with the other one as a radius, describe arches intersecting each other, and the point of section will be the fourth angle of the rectangle, to which the two remaining sides may be drawn from the extremities of the others. An oblique parallelogram may be drawn exactly in the same manner, only an angle of it must be given as well as the sides; and the two sides must be placed so as to make an angle equal to this angle, and then the remainder of the construction is exactly the same as that of a rectangle.

It need hardly be mentioned, because it is apparent without any description, that the same data which suffice for constructing one square, rectangle, or parallelogram, must suffice also for constructing another in any other place or at any other time; and that, if the data, that is to say, the one side in the case of the square, the two sides in the case of the rectangle, and the two sides and the included angle in the case of the parallelogram, are exactly equal in any two instances, then the proof of the equality in every respect of the figures themselves, is as clearly established as it can possibly be, even though the one of them were applied to the other and observed to coincide with it. And we may repeat, that this is the useful proof in practice; and that the coincidence, how pleasant soever it may be for closet mathematicians, is of little avail in real life. It is desirable, for instance, that an acre of land should be the same quantity or extent of surface in Cumberland as in Cornwall; and that a mile of the road between London and Bath should be exactly the same length as a mile of that between Edinburgh and Glasgow; but it would puzzle all the geometers that ever lived to bring the two acres of land in the different counties, or the two miles in length of the different roads, to any comparison except through the medium of some measure, or other means of connexion, which had nothing to do with the applying of the one of them to the other, and observing whether they coincided or not.

The next step of our progress, in examining the relations of figures, and of the boundaries and other parts of figures, is naturally that which involves some medium of comparison; and for this purpose we must have recourse to the doctrine of ratios, as explained in a former section; because equality of ratios is very often the only equality of which we can avail ourselves, in such comparisons. But before we can enter upon this with the proper advantage, it is necessary that we should understand a little more about the relations of the different parts of the same figure, both to each other and to the whole.

All the figures with which we are concerned in plane geometry, may be reduced to these four classes:—First, circles; secondly, squares; thirdly, triangles; and fourthly, rectilineal figures having more than three sides, of which last squares are a particular division; but they are so simple as compared with the others, that they require to be separated.

The general condition which figures must have, in order that we may apply the doctrine of ratios to them, and reason from the one to the other, or from parts of one to other parts of the whole of the same, is the property of their being

SIMILAR FIGURES.

1. Figures are said to be similar, when they are exactly of the same shape, but different from each other in size. Equal sized figures of the same shape are of course similar, as well as unequal sized ones; but they are equal as well as similar; and any ratios which they, or corresponding parts of them, may have to each other, are ratios of equality, not merely in the relation itself upon which the ratio is founded, but in the

terms of the ratio; and it is evident, that from the comparison of two ratios of this kind, no useful conclusion can be drawn. Thus, if a ratio were stated

$$12 : 6 = 12 :,$$

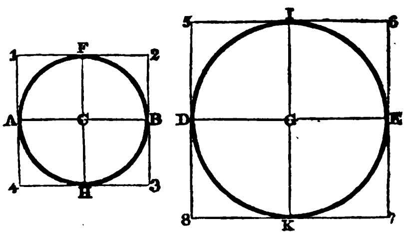
and the fourth term sought by an operation in the common rule of three, the fourth term would be the same as the second, or 6 again, and we should learn nothing by the operation. This is what is called an identical proposition: and it very often happens that those who have not disciplined their minds in the practice of reasoning in a close and logical manner, make use of such propositions in succession, or "reason in a circle," as it is called, that is, go round like a mill horse, and end just where they began,—or rather their saying, for it is not argument, like the circumference of a circle, has neither beginning nor end; is founded on nothing and leads to nothing.

Thus, the species of similarity which is useful to us, is similarity without sameness. And there are three conditions requisite, in order to establish the similarity of any two plane figures. First, they must have the same number of sides in each, and, by necessary consequence, the same number of angles. Secondly, all the angles of the one must be each to each equal to all the angles of the other, when taken in the same order of succession; and thirdly, the sides about the equal angles, and which are situated in the same order with regard to the angles, must be proportionals. This third condition is, in some figures, so closely connected with the second one, that the one of them cannot exist without the other; and therefore it might, perhaps, be possible to deduce the one from the other, at least in the case of one or two figures; but still it is much better to take both parts into the definition, as then it is rendered perfectly general.

2. *Circles* have only one side each, and they have no angles;

therefore there is nothing connected with one circle, which can affect its similarity to every other circle; and consequently all circles are similar figures, and may be reasoned upon as such.

A square has four sides and four angles, and they are all equal to each other in every square, so that one square cannot be dissimilar to another, without ceasing to be a square, and so falling into another class of figures.



Let there be any two circles, of which C and G are respectively the centres, and A B and D E the diameters; also let 1, 2, 3, 4, and 5, 6, 7, 8, be two squares, of which the sides are respectively equal to the diameters of the circles; the circles are perfectly similar, and so are the squares; and the diameters of the circles are respectively equal to the sides of the squares. Therefore,

Circle C : circle G = square 1, 2, 3, 4 : square 5, 6, 7, 8.

That this must be the case, is perfectly evident; for whatever may be the ratio of the circle to the square in either case (and that is not our present business), the two circles must stand to their own diameters in the same ratio; and therefore they must stand to each other in some ratio of those diameters. The diameters themselves have no property but their lengths; and we cannot obtain the ratios of surfaces without products of length and breadth. But the circles have exactly the same ratios to the cross diameters F H and I K that they have to A B and D E; because in the same circle all diameters are, by the very construction of the circle, equal to each other. Therefore the circle C is to the circle G, as A B to D E, and also as F H to I K. But A B = F H and D E = I K; therefore compounding the ratios, we have,

$$\text{Circle C} : \text{circle G} = AB \times FH : DE \times IK.$$

But $AB = FH$, and $DE = IK$;

Therefore,

$$\text{Circle C} : \text{circle G} = AB^2 : DE^2; \text{ that is,—}$$

Circles are to each other in the ratio of the squares of their diameters; that is to say, if the diameter of one circle is double that of another, the area of the circle will be four times; if triple, nine times; and so on in the case of every other multiplier, whether greater or less than the number 1.

But the circumferences of the two circles have evidently, in each circle, the same relation to the diameter; and though curves, and not straight lines, the circumferences still are lines, not surfaces; and therefore their ratio must always be a simple ratio; but each circumference stands in the very same relation to its diameter as the other does. Wherefore, the circumferences of circles have the same ratio as their diameters; and the surfaces of two different circles are to each other in the ratio of the squares of their circumferences, as well as of their diameters.

It would be easy, in like manner, to show that any two lines similarly placed in two circles, have the same ratio as the diameters or circumferences of those circles, but it is not necessary, and indeed we might have come at once to all the conclusions which have been stated, from the simple fact, that no element of dissimilarity can affect circles, or corresponding portions of circles.

There is one practical application of the equal ratios of the diameters and circumferences of all circles which it is useful to bear in mind. If we know the diameter, the circumference, and the content of any one circle; and the diameter, the circumference, and the content of another, we are enabled, from the ratio, to find the other two particulars for the other one. The approximate dimensions and area of one circle are as follows:

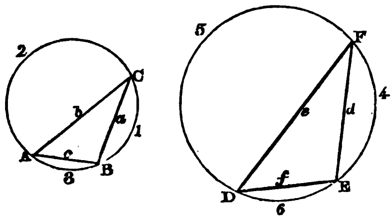
Diameter $= 1\cdot$
Circumference $= 3\cdot1416$,
Area $= \cdot7854$;

and from these it is easy to find the particulars of other circles.

If we turn back and examine the squares in which the circles are inscribed, we find that their relation is a very simple one; for, from the very meaning of square, it is obvious that the areas of all squares are to each other in the ratios of the squares of their sides. It is worth while, however, to look at the boundaries of the squares,—or, as they are sometimes termed, their perimeters. In each square the perimeter is four times the side; and therefore the perimeter has always the same ratio to the perimeter of another square, as the side has to the side.

3. *Triangles* form our next subjects of comparison; and the first comparison to be made is that of triangles which are equiangular, or have all their angles respectively equal each to each; and the object is to show that the sides about the equal angles are directly proportional.

For this purpose, let A B C and D E F be two triangles which are equiangular, that is, which have the angle A of the one equal to the angle D of the other; the angle B of the one equal to the angle E of the other; and the angle C of the one equal to the angle F of the other; then the sides about the equal angles are directly proportional.



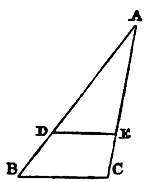
[When comparing the sides and angles of triangles, it is very convenient to mark each side with the same letter as the angle to which it is opposite, only to mark it in a different character, italics for instance, while the angles are marked with capitals: thus *a*, *b*, and *c*, are respectively the sides opposite the angles A, B, and C, in the one of the above triangles; and *d*, *e*, and *f*, are, respectively, the sides opposite the angles D, E, and F, in the other. This is not a matter of necessity; but it is one of considerable advantage.]

Describe, as already explained, a circle about each triangle, so as to pass through the three angular points, and the angles of each triangle, will be angles at the circumference, standing upon arcs of their respective circles; namely, in the triangle A B C the angle A stands on the arc 1, the angle B on the arc 2, and the angle C on the arc 3; and in the triangle D E F the angle D stands on the arc 4, the angle E on the arc 5, and the angle F on the arc 6. So that in each circle the three angles of the triangle, taken altogether, stand on the whole circumference. But as the angle A of the one is equal to D of the other, the arc 1 of the one must be equal to the arc 4 in the other; and as the side *a* of the one, and the side *d* of the other, are the chords of equal arcs, they stand in the same relation or ratio to their respective circles. For similar reasons, the side *b* and the side *e* stand in the same relation to their respective circles; and so do the side *c* of the one, and the side f of the other. But if their relations to the circles are equal, then the lines themselves must stand to each other in the ratios of their own lengths; that is, a to b as d to e, a to c as d to f, b to c as e to f; and so on in every other case in which the corresponding sides about an equal angle in the two triangles, can be compared with each other. Therefore, if triangles are equiangular, the sides which contain equal angles are directly proportional, the sides opposite the other equal angles being the corresponding terms, that is, both the antecedents or both the consequents of the ratios. Thus, the sides about the equal angles A and D, are either

$$b : c = e : f, \text{ or } c : b = f : e;$$

and as these are direct proportionals, they will remain proportionals under all the changes of which proportionals are capable.

It follows immediately from this, that if two sides of a triangle are intersected by a line parallel to the third side, the part cut off is similar to the whole triangle; and the segments of the two sides, on both sides of the dividing line, are proportionals, and have the same ratio as the sides which are divided.



Let A B C be any triangle; and let the line D E be drawn parallel to B C, meeting the sides A B and A C in the points D and E. The triangle A D E is similar to the whole triangle A B C; A D is to A E, and also D B to E C, as A B is to A C. For the angle A is common to the whole triangle and the part; and the angle D, exterior on the parallels, is equal to the angle B interior and opposite, and the angle E is, for the same reason, equal to the angle C; therefore the whole triangle A B C and the part A D E are equiangular, and the sides about the equal angles directly proportional; that is, A D : A E = A B : A C; but B D and C E are the differences of those proportions, and therefore, by separation, they have the same ratio, or D B : E C = A B : A C. These also being directly proportional, admit of all the changes of which such proportionals are capable.

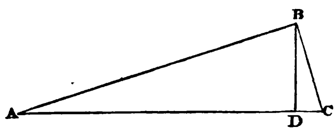
By means of this last principle we are enabled to divide any given line in any proposed ratio; and also to find a fourth proportional to three given lines, or perform an operation in the rule of three by means of lines alone.

Thus to divide a given line in any ratio, we have only to draw a line at one extremity of it, making any angle, and divided into the two terms of the ratio; and then, by joining the extremities of the lines, and drawing a parallel through the point which separates the known terms, we have the given line divided in the same ratio. This is not confined to division into two parts; but if one line is given, which is divided into any number of parts in a ratio, and it is required to divide a line of a different length into the same number of parts, having the same ratios, the principle will apply; and in practice it often saves a great deal of trouble.

In the finding of a fourth proportional, we have nothing more to do, than to draw two lines, making an angle, to set off the first and second terms, one upon each of them, and join by a line, then set off the third on the same line with the first, draw a parallel, and that parallel would cut off the fourth proportional from the same line on which the second was set off. These are so simple, that it is unnecessary to illustrate them by any diagrams or examples; we shall therefore proceed to another problem, in which we require the assistance of similar triangles. This problem will require allusion to a principle somewhat different.

4. In a right-angled triangle, if a perpendicular is drawn from the right angle to the opposite side, the triangle is divided into two triangles, which are similar to the whole, and to each other; that perpendicular is a mean proportional between the segments into which it divides the base, — the greater segment being on the same side of the perpendicular with the greater of the two remaining sides of the triangle.

Let A B C be a right-angled triangle, having the right angle at B, and let a perpendicular B D, be drawn from the right angle B to the opposite side A C; the two triangles A D B, and B D C, into which the triangle A B C is divided by this line, are equiangular, and therefore similar; and the perpendicular B D is a mean proportional between the two segments A D and D C, the greater segment A D being on the same side with, or adjacent to the greater side A B of the original triangle, and the less segment D C, on the same side with, or adjacent to the less side B C of the original triangle.



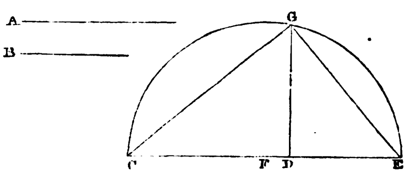
Because B D is perpendicular to A C, the angles B D A, and C D B at D are equal to each other, and each of them a right angle. But the angle at B is a right angle, and therefore the two parts of it D B C and D B A are together equal to a right angle. But in the triangle B D C the angles D B C, B C D are together equal to a right angle. Consequently, A B D is equal to B C D, and D C B equal to B A D, therefore the three triangles

A B C, A D C, and B D C, have the angles opposite A C, A B, and B C all right angles; the angles opposite A B in A B C, A D in A D B, and B D in B D C, all equal to the angle at A, and therefore equal to each other; consequently the remaining angle in all the three triangles is also equal, and the triangles are similar, and the sides about the equal angles directly proportional; and among others the greater side about the right angle, is to the less in the one as the greater side about the right angle is to the less in the other. But A D is the greater, and D B the less in the triangle A D B; and D B is the greater, and D C the less about the right angle in the triangle at B D C. Therefore

$$AD : DB = DB : DC;$$

wherefore D B the perpendicular is a mean proportional between the two segments A D and D C.

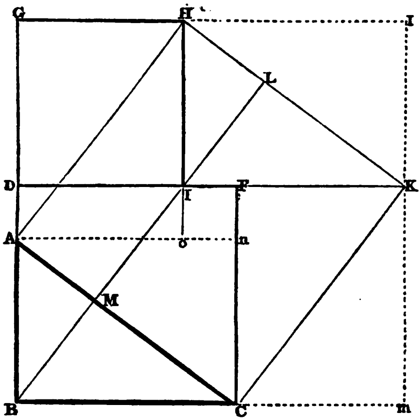
We are now in a condition for obtaining a solution of the problem to which we alluded: To find a mean proportional between two given straight lines. Let A and B be any two straight lines, it is required to find another straight line which shall be a mean proportional between them; take any straight line, and from it cut off C D equal to the given line A, and immediately conterminous with it, or beginning where it ends, take the part D E equal to the given line B. Bisect the whole C E in F; on F as a centre, with radius C E or F E, describe a semicircle. At the point D draw a perpendicular to C E, and produce it till it meet the circumference of the semicircle in G; and D G is a mean proportional between C D and D E, or between their equals, the given lines A and B.



Join C G and E G; and because C G E is an angle in a semicircle, therefore it is a right angle, and C G E is a right-angled triangle, and G D the perpendicular from the right angle upon the opposite side; therefore G D is a mean proportional between C D and D E, the segments of the side opposite the right angle, which segments are respectively equal to the given lines A and B, between which the mean proportional was required.

The given lines A and B might have been the containing sides of a rectangle; and if it had been required to find a square equal in area to the rectangle, the proceeding would have been exactly the same as that just mentioned; namely, if the two containing sides had been placed conterminously on the straight line, a semicircle described on the sum of their lengths as diameter, and a perpendicular drawn from the point where they met to the circumference of the semicircle, this perpendicular would have been the side of the square, equal in area to the rectangle; and the side is the only datum requisite for constructing the square itself.

There are some applications of this and the preceding principles, to figures containing more than three sides, and not regular, which might have been introduced here; but there is one general truth, together with some collateral or resulting ones, with which it is desirable to be acquainted, before we proceed to these; and as this truth is a very important one, as one of the main links by which geometry and the arts of calculation by quantities generally, or by numbers practically, are connected together, we shall consider it a little in detail. For this purpose we introduce the following diagram: and as it is considerably more complicated than most of those which we have previously introduced, we have endeavoured to throw a little graphic effect into it, in order that the reader may the better understand it; and we shall go over the different parts of the diagram, before we point out the use to which it is to be applied.



Upon examining it, the reader will perceive that there are in it lines of four different degrees of strength.

First, in the reader's left-hand corner, at the bottom, a right angled triangle, A B C, of which the lines are more boldly drawn than any of the rest. Then the left hand portion is occupied by two squares, not so boldly drawn as the triangle, but bolder than the rest of the diagram. The undermost of these squares is the largest, and it will at once be perceived, that it is the square upon B C, the greater of the two sides which contain the right angle in the right-angled triangle A B C. Above this, there is a smaller square, having D, G, H and I at the four corners; and though this square does not stand upon A B, the less of the two sides which contain the right angle B in the triangle A B C, yet the sides of it are made equal to A B, and therefore it is the square of A B with the same truth as though A B itself had formed one of the sides.

Secondly, a larger square, with one angle at the left, one at the top, one at the right, and one at the bottom, where the letters A, H, K, and C, are situated, may be traced, having A C for one of its sides, and the remaining sides, A H, H K, and K C, marked in much fainter lines; and it will be seen that this is the square upon A C the side of the right-angled triangle, which subtends or is opposite to the right angle at B, which side is called the *hypotenuse*, because it "holds under" or is opposite to a right angle; and in consequence of this has a property which no side of a triangle, not having a right angle, can possibly possess.

Thirdly, it will be observed, that a faint line is drawn from the right angle B through M, I, and L, parallel to both sides of the square upon A C, and therefore crossing A C, and meeting H K at right angles. Of course the part M L divides the square of A C into two rectangles, each having A C or its equal for one of its containing sides, and one of the segments into which A C is divided for the other.

Fourthly, there will be observed dotted lines at the top from H to *i*, on the right from *i* through K to *m*, and at the bottom, from *m* to C, which complete the whole diagram as a square. Also, from A through *o* to *n* there is a dotted line, equal and parallel to B C; and from I to *o* there is a line, equal and parallel to D A, or to F *n*; and F and K are joined by a faint line, which is a continuance of D F.

It will be seen that by means of these dotted lines, the entire square G *l m* B is divided into four equal rectangles, A B C *n*, C A *o* H, H I K *l*, and C F K *m*; and that these four rectangles are each equal to the rectangle A B × B C, or double the area of the triangle A B C. Besides this, there is the small square, I F *n o*.

in the centre of the diagram; and because D B is equal to B C, D A is the difference of A B and B C, and as I *o* the side of this small square is equal to D A, this small square is the square of the difference between A B and B C. But as B D or A G is equal to B C, B G is the sum of A B and B C, and B G is the side of the whole square of the diagram; and the diagram is the square of the sum of A B and B C. Therefore, from mere inspection of the diagram, we have this general conclusion:

The square of the sum of two lines, is equal to four times the rectangle under those two lines, together with the square of their difference.

Upon carefully examining the figure, it will be found that, whatever is the position of A C, the four triangles external of the square of A C, but contained in the square of the sum of A B and B C, must be each equal to the given triangle, and therefore altogether four times that triangle, or twice the rectangle A B × B C. So also the square upon A C must, in all cases, be equal to four times the triangle, together with the square of the difference of the sides.

If one were to imagine the square upon A C to be turned toward the right, the angle A of that square would gradually approach the point D, and as it did so, A B would approach in length to B C. If they became equal, A D would vanish, because the lines would have no difference, and the small square I *o n* F, would also vanish, and the square upon A C would become equal to four times the triangle, or twice the rectangle of the equal sides about the right angle, that is, half the square of their sum.

On the other hand, if the square upon A C were supposed to be turned round toward the left, so that A B shortened by the point A approaching the point B, then the small square I *o n* F would increase, and the square upon A C would also increase in respect to the area of the triangle; and if we considered A B to be shortened till it vanished by the point A coinciding with the point B, then the square of A C, the square of B C, and the square I O N F would all be equal and coincident with each other, and the triangle would, at the same time, have vanished.

Hence, we can see the limits between which a right-angled triangle can exist. The one limit is the equality of the sides about the right angle, and in this case the square on the hypotenuse is equal to four times the area of the triangle. The other limit is that at which the triangle vanishes, by one side becoming equal to the hypotenuse, and there being, of course, no third side. Between those limits, there may be any degree of inequality between the sides A B and B C, and whatever their lengths are, the square of the hypotenuse is always equal to four times the area of the triangle, together with the square of the difference of the two sides which contain the right angle.

These, however, are not exactly the principles which we are seeking to establish; and we have merely mentioned them incidentally, in order that our examination of the diagram might not be altogether without a result. The principle is as follows:

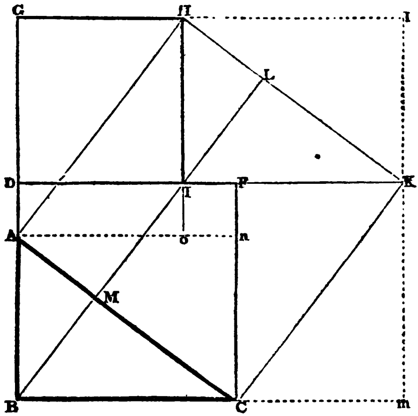
In any right-angled triangle, the square of the hypotenuse or side subtending the right angle, is equal to the sum of the squares on the sides which contain the right angle.

We repeat the diagram, in order that the reader may not have to refer to it over several pages.

A B C is any triangle, right angled at B; A C is the hypotenuse or side opposite the right angle; and it has to be shown that the square upon A C is equal to the two squares upon A B and B C, that is, that

$$AC^2 = AB^2 + BC^2.$$

Produce A B in the direction of A: take B D = B C, complete the square D F C B, and it is the square upon B C. In B D con-



tinued beyond D, take D G equal to A B, and in D F take D I also equal to D G, and D G, D I, are sides of a square. For, as the angle B D F, being the angle of a square, is a right angle, the adjacent angle G D I is also a right angle. Complete the square with G H and I H both equal to D G, and G D I H is equal to the square upon A B; for D G was made equal to A B, and all the other sides equal to D G, and the angle G D I is a right angle. Thus we have obtained the squares upon A B and B C.

Join A H, and A H is equal to A C, and at right angles to it. For, comparing the two triangles H G A and A B C, we have, H G = A B, and G A = B C, and the contained angle in each case a right angle; therefore A H is equal to A C. But the angle G A H is equal to B C A, for they are opposite equal sides in equal triangles; and the two angles G A H and B A C, at the point A, are together equal to one right angle, for they are equal to the two smaller angles of a right-angled-triangle. But A is a point in the straight line B G, and all the angles at that point on one side of the line are equal to two right angles. Two of them, namely, B A C and G A H, are together equal to one right angle; and, therefore, C A H, the supplement, must be a right angle. Consequently, C A and A H are two sides of a square. Complete this square by C K and H K, each equal to A C or A H; and A H K C is a square, and it is the square upon A C, the hypotenuse of the triangle.

Our next business is to show that this square is equal to the two former squares, namely, those upon A B and B C. Join F K, and because H I F is a right angle, H K equal to A C, H I equal to A B, and the contained angle at H equal to the contained angle at A, F K is in the same straight line with I F, and the triangle H I K is equal to the triangle A B C. But H I K is contained in the square of the hypotenuse, and not in the squares of the sides; and A B C its equal is contained in the squares of the sides, and not in the square of the hypotenuse. Therefore, those two equal triangles increase equally the one square in the one case, and the two squares in the other; and, therefore, whether they are supposed to be taken away or supposed to remain, they do not make any difference between the compared quantities.

The triangle H G A, toward the left hand at the top, has been already shown to be equal to A B C, and it is wholly contained in the two squares of the sides, and not in the square of the hypotenuse. But the triangle K F C has K C equal to A C, F C equal to C B, and F K equal to A B; therefore it is also equal to A B C, and consequently to H G A; and the triangle K F C is contained wholly within the square of A C, and its equal H G A wholly within the squares of A B and B C. Therefore, if we take away or leave those equals, one of which belongs to each of the quantities compared, we shall not make any difference. Take away the two triangles which are contained in the squares of A B and B C, and the two equal triangles which are contained in the square of A C; and by taking equals no difference is taken away. But the two squares in the one case, and the one square in the other, are reduced to the same identical space, namely, the irregular space contained by the lines H A, A C, C F, F I, and I H; and we need not add, that this identical space must be equal to itself. Therefore, the square of A C is equal to the sum of the squares of A B and B C. A B C is any right-angled triangle whatever; wherefore the square of the hypotenuse of every right-angled triangle is equal to the sum of the squares of the sides containing the right angle.

Another demonstration of this truth might be derived from the same figure. For, upon examination, it will be perceived, that the square upon the side opposite the right angle is divided into two rectangles, A H L M, and M L K C, by the line B M L drawn from the right angle of the given triangle parallel to the sides of the said square on the side opposite that angle; and if each of these rectangles is compared with the square of the side adjacent to it, and about or containing the right angle, it will be seen that there are in the diagram two oblique parallelograms, A H I B and B I K C, one of which is upon the same base and between the same parallels, both with the rectangular portion of the larger square, and with the square of the adjacent side. Therefore, the rectangle, and the square on the adjacent side, are each equal to the oblique parallelogram, which is on the same base, and between the same parallels with each of them, though it is between the same parallels with them in its different dimensions; consequently, the two parallelograms into which the square on the side opposite to the right angle, and the two squares upon the sides containing the right angle, are equal to each other on the adjacent sides, that is to say, the greater rectangle is equal to the square on the greater side, and the less rectangle is equal to the square on the less side. But the two rectangles taken together make up the square upon the side opposite the right angle; and therefore their sum is equal to the sum of their equals, namely, that of the squares upon the two sides containing the right angle. In other words, the square on the side opposite the right angle, is equal to the sum of the squares on the sides containing that angle as before.

If the angles of a right-angled triangle are expressed by the letters A, B, C,—and it is not necessary to draw the triangle, as we are speaking of a right-angled triangle generally, without any reference to the particular lengths of its sides, but merely of the relation of their squares to each other, which applies to every possible right-angled triangle,—we may form a perfect notion of the triangle and its properties, without any reference to a particular triangle actually drawn, by merely referring to the angles by the letters A, B, and C, B being the expression of the right angle.

So, also, if we call the sides by the small letters corresponding to the capitals which express the angles opposite to them, we shall have a, b, and c, as a general expression for the sides of any right-angled triangle, b being the side opposite to the right angle, and a and c the two sides about or containing the right angle.

Taken in terms of the sides, as expressed by letters, the algebraical notation of the above truth is,

$$b^2 = a^2 + c^2,$$

in which expression it is of no consequence whether a or b indicates the greater of the sides about or containing the right angle.

It will be easily perceived from this, that if any two of these three quantities, that is, the sides b, a, and c are given, the remaining one can be found by an arithmetical operation. If the sides about the right angle are given, then the expression for the hypotenuse in terms of the sides, is $\sqrt{(a^2 + c^2)} = b$; and the expression for any side in the case of the hypotenuse being given, is the square root of the difference between the square of the hypotenuse and that of the given side. Thus if a is the given side, the expression for the required side c is $\sqrt{(b^2 - a^2)} = c$.

One of the cases of the application of this principle, which occurs most readily in practice, is that in which one side of the right-angled triangle is a distance measured on the level ground, and another the height of an upright object; as, for instance, a wall, a tree, or a steeple; and then the hypotenuse, or longest or slanting side of their angle, is the line extending from the farther extremity of the distance measured upon the ground, to the top of the upright object. For the sake of distinction, the line measured on the ground is called the base, and the height of the object the perpendicular, while the remaining side retains the name of hypotenuse. Thus, for instance, if the width of a street were 80 feet, and the height of a house rising perpendicularly from it 60 feet, and if it were required to find the length of a line which would extend from the opposite side of the street to the top of the house, the expression for the length of this line would be,

$$\sqrt{(80^2 + 60^2)} = \sqrt{(6400 + 3600)} = \sqrt{10000} = 100,$$

so that the line reaching from the opposite side of the street to the top of the house would be 100 feet.

5. From the complete establishment of the relation between products and rectangles (the product or the rectangle being a square in the case of two equal factors) which results from the principles that we have now stated; and from the constant ratio of equality between the square of the hypotenuse of a right-angled triangle, and the sum of the squares of the two sides containing the right angle; we are enabled to substitute either quantities generally as expressed by algebraical notation, or numbers as measures in particular cases, in place of lines,

so that the products of those general quantities express rectangles generally, the products of equal factors express squares, and the numerical products of the lines expressed in known measures, give us the areas of rectangles or squares, in squares of the same measure. It is, however, to be understood, that though, in the particular cases, the product, whether square or rectangle, is a number, as well as the sides, it is not a number of the same kind with them; for in all cases it means surfaces, the unit meaning a square, the side of which is the unit in the line. Thus, for instance, if a piece of board is 5 feet long and 3 feet broad, the area of the board, considered as a rectangle is 5×3 feet, that is 15 feet; but these 15 are not of the same kind, either with the 5 or the 3; they are square measures or surfaces, having length and breadth, situated at right angles to each other, and of equal measure, whereas the 5 and the 3 are merely lengths, or of one dimension only, and express the sides of the squares in which the area is represented.

In considering this relation between surfaces and products of two factors which stand to each other in the relation of length and breadth, it is not absolutely necessary that the two factors should be in the same denomination, in measure of length, provided that we know exactly the proportion which the length of one of them bears to that of the other. Thus, for instance, if one dimension is expressed in yards, and another in feet, of which there are three in a yard; then the product will neither be square yards nor square feet, but every unit in it will be a rectangle, the length of which is a yard, and the breadth a foot; and it will take three such rectangles to make up a square yard, or each of them will evidently make three square feet. In all cases of the multiplication of two factors of different denominations, the relation between which is known, it is very easy to reduce or change the product to squares of either factor; for we have only to multiply it by the number of times that the greater factor is contained in the less, in order to get squares of the less factor, or to divide it by the number of times that the greater contains the less, in order to get squares of the greater factor. Thus, for instance, if one factor were feet, and the other inches, we could reduce the product to square inches by multiplying it by 12, the number of inches in a foot; or we could reduce it to square feet by dividing by 12, the number of inches in a foot. There is, of course, nothing peculiar in the mere fact of the dimensions in this case being feet and inches; for if they were any other measures whatever, exactly the same principles would apply, and we should have nothing more to do than to multiply by the number of the less which the greater contained, in order to get squares of the less, or to divide by the same, in order to get squares of the greater.

As in the case of the product of two quantities or numbers, it is of no consequence, that is, produces no effect upon the value, though the factors are made to change places by the multiplier being made multiplicand, and the multiplicand multiplier; as, for instance, a and b being any quantities whatever, and consequently the representatives of all quantities, the product $a\,b$ is exactly equal to the product $b\,a$, and the one might in any case be substituted for the other; so, when a rectangle is expressed by the product of its length and breadth as factors, it makes no alteration in the value, or even the form of the rectangle, whether the one factor or the other is considered as length, provided that the other one, which is understood to stand at right angles to that one in the figure, is considered as breadth. Thus, if the length and breadth of any rectangle, are represented by two lines A B, and B C, which of course mean any lines whatsoever, because neither of them is restricted to any measure in length expressed or expressible by a number, the area of the rectangle is either A B × B C, or B C × A B, which products are perfectly identical, and either of them might be with the most perfect accuracy substituted for the other; so also if the base and perpendicular of a triangle, which are the length and breadth of the rectangle, which is exactly double the triangle, are represented generally by the algebraical expressions a and b, or geometrically by two lines A B and B C, then the expressions $\frac{ab}{2}$, $\frac{ba}{2}$, $\frac{AB \times BC}{2}$, $\frac{BC \times AB}{2}$, all equally represent the area of the triangle; and to apply them to any case, it is only necessary to state the value of a and b in numbers, or those of A B and B C in lines, to obtain the practical expression which correctly represents the value of that particuticular triangle as expressed in numbers, or by measures, it being generally understood that the measures of the two lines are of the same denomination, so that the product without any reduction represents squares of the lineal measure, which, however, is not absolutely necessary as affecting the general principle; because the product of unequal denominations can be reduced to squares of either of the two, by multiplying or dividing by the ratio of the greater to the less, as was explained in the preceding part of this article.

6. By means of this perfect correspondence between rectangles and the products of numbers expressing the lengths and breadths, that is, the containing sides of those rectangles, we are enabled to compare surfaces, and determine their equality or inequality in a far more simple manner than can be done by geometry alone; and though this mode of comparison through the medium of numbers is not so direct, or immediately apparent to the eye as the geometrical comparison of the rectangles themselves, actually drawn in a diagram, and palpable to the eye, yet it possesses other advantages over the geometrical

method, in addition to the fact of its being more general, and more simple. It enables us to the carry the geometrical truth with equal force and clearness into those subjects which do not admit of geometrical representation. For instance, it enables us to examine compound ratios, or those which result from the multiplication of the corresponding terms of given ratios; and to do this, not only in the case of two ratios, which is as far as the elements of plane geometry can carry us, in as much as those elements involve no higher power than a surface, but farther even than the elements of solid geometry can carry us; for these do not extend beyond a solid, which is the product of three factors standing to each other in the relation of length, breadth, and thickness. If we make use of a fourth factor, or a third multiplication, the result which we obtain is taken out of the class of geometrical quantities altogether; and can have no meaning, unless we consider three of the factors as length, breadth, and thickness, and the fourth factor as a number multiplying the solid of which they are the dimensions; or two of the factors as length and breadth, and the product of the other two as multiplying the surface of which these two are the dimensions; or one of the factors as a line, and the product of the other three as multiplying the length of that line.

But when we are released from the restraint which the geometrical consideration imposes upon us, by tying us down to the single factor as a line, the two factors as a surface, and the three factors as a solid, we are enabled to institute comparisons by means of any number of factors or of multiplications that we please; and the relations of the products of those factors are perfectly understood, and equally true, whether the factors themselves are or are not quantities which it is possible to express by lines. This generalisation of the property of a rectangle, of being always expressible by the product of the length and breadth, in the extending of it to all quantities whatsoever, of which the values can be expressed either by single expressions, or by the relations of other expressions, which is the only means of accurately expressing in commensurable quantities—this generalisation is one of the most useful in the whole range of the mathematical sciences; and it is the one which enables us to bring the force of geometrical truth, and the clearness of geometrical conviction into the business of common life, and into every speculation in knowledge, and every plan in any occupation whatsoever, to which reasoning can be made to apply. It has also this additional advantage, that how far soever we may carry our reasoning by analogy, our analogies, in order to be sound in their progress, and true in their ultimate conclusions, must be geometrical in every individual step, how much soever the results may be complicated, beyond the three dimensions, which, as we have said, and as every one must perceive, form the ultimate limit beyond which no purely geometrical magnitudes, as a real and separate existence, which can be set apart and contemplated, as a subject of thought, in itself and without necessary allusion or analogy to other subjects, can extend.

Thus, for instance, every change which we can by possibility produce upon any quantity or existence whatsoever, can always be referred to an increase or a decrease of it; and in the case of known quantities, both the increase and the decrease may, in every case, be considered as the results of multiplications; for, if there is something added, this addition must be of known value in terms of that to which it is added, before we can express the sum; and therefore the sum will always be the original quantity multiplied by 1 + the ratio of the addition to the said original quantity. In like manner, if the difference is a subtraction, the remainder can always be expressed by the product of the original quantity multiplied by 1—the ratio of the subtracted part to the original quantity.

By viewing the matter in this light, we get a clear and distinct notion of the two parts of which every result must be made up; namely, the quantity, and the operation, or as we express it in matters of common business, the materials, and the work done. The last, the work done, or the operation performed, whatever it may be in the result, is never in itself a geometrical quantity, because action, of whatever kind it may be, cannot be in itself a line, a surface, an angle, or a solid; and as, even in the most rigid geometry, it is impossible to imagine the very simplest problem to be performed, or the most obvious result to be arrived at, without the performance of some degree of some kind of action, the geometer is forced tacitly to assume the fact of this performance; and this assumption of a necessary element, which from its very nature does not admit of geometrical definition, is the line by which geometrical reasoning, and common reasoning in the business of life are cut asunder, and rendered in a great measure inapplicable and useless to each other; instead of being, as they are in truth, the two pillars which support the mind in every judgment which it can form, and the two hands, as it were, by means of which it grasps, conducts, and accomplishes every object of which it has previously taken the proper view, and seen that the accomplishment is possible, and the data sufficient for arriving at it.

In cases of direct multiplication and division, we are brought immediately to the geometrical correspondence of the two factors to the length and breadth of a rectangle, and the product as the area or surface of that rectangle; and as each multiplication, in the case of any number of factors, however numerous or however unlike each other they may be, is still nothing but the multiplication of two factors, each step is reduced to the simplicity of the geometrical truth. It is the same in cases of division, how often soever we divide; for a division by any number or quantity produces exactly the same result as multiplying by the reciprocal of that quantity, that is, by 1 divided by the divisor; or, if we call the divisor a, which may represent every possible number and every imaginable quantity, whether expressible by numbers or not, and call the dividend which is to be divided by it b, then the quotient will be $= b \times \frac{1}{a}$, which is nothing more than once b divided by a, or simply b divided by a, that is, $\frac{b}{a}$, the fractional expression for a quotient as it has been previously explained.

7. The general relation which we have endeavoured to explain in the preceding article, and which ought to be duly studied and worked out by every one who wishes to have a knowledge of mathematics, whether general, geometrical, or arithmetical, enables us to simplify greatly all our reasonings concerning figures which are similar but not equal, and those which are equal but not similar; and the first and simplest application is that to figures described on the whole and the several parts of a line, or on a line equal to the sum of two or more lines, and also on those lines taken singly or together in any order.

Upon looking back to the diagram in article 4, which enabled us to show that the square upon the hypotenuse of a right-angled triangle, is, in all cases, exactly equal to the sum of the squares on the two sides containing the right angle, it will be be seen that the square of the sum of the sides containing the right angle is equal to four times the rectangle or product of those sides, together with the square of the difference. But the triangle in that diagram is any right-angled triangle whatsoever, consequently the two sides containing the right angle are any two lines whatsoever, and may be represented by two letters, a and b for the general expression, or by numbers for any given particular case. Therefore, we may state generally, that the square of the sum of two quantities is equal to four times the product of the quantities, together with the square of their difference.

Thus if a is the greater quantity, b the less, and d the difference, the expression is,

$$(a+c)^2=4\,a\,c+d^2;$$

for as d is the difference, then $a=c+d$, or $a+c=2\,c+d$; consequently $(a+c)^2=(2\,c+d)^2$.

If we express the sum of the two quantities in terms of c and a, it is evidently $2\,c+a$; for the sum is equal to twice the less together with the difference. By squaring this, we obtain,

$$4\,c^2+4\,c\,d+d^2,$$

the first and second terms of which express four times the rectangle, or product of the given quantities a and c; for

$$a=c+d$$

Multiply by c

and . . . . $a\,c=c^2+c\,d$; consequently, four times $a\,c$ is . . . . $=4\,c^2+4\,c\,d$, to which add d^2; and the sum is, . . $4\,c^2+4\,c\,d+d^2$ as before.

If we had taken the expression for c, in terms of d, by making it $a-d$, then the sum of the two quantities would have been expressed by $2\,a-d$; and the square of it by

$$4\,a^2-4\,a\,d+d^2;$$

the first and second terms of which are of exactly the same value as those of the former expression, in which the whole value was taken in terms of c; and if we leave out of both the common term d^2, the leaving out of which cannot produce any difference, then we have,

$$4c^2+4cd=4a^2-4ad.$$

Hence, four times the square of the less quantity, together with four times the product between the less quantity and the difference, is equal to four times the rectangle or product of the quantities; and four times the square of the greater, wanting four times the rectangle or product of the greater and the difference, is also equal to four times the rectangle, or product of the two quantities.

8. Upon again looking back at the same diagram, it will be manifest from mere inspection that, whatever the lengths of the two sides A B and B C including the right angle may be, the four angles of the square upon the hypotenuse must always be upon the sides of the larger square, which is constructed upon the sum of the other two sides of the triangle, and must divide each of the sides of that square into two parts exactly equal to the sides about the right angle. Also it will be seen that the parts of the larger square situated external of the square of the hypotenuse, must in every case be equal to four times the area of the right-angled triangle, or to twice the product or rectangle of the sides A B and B C, which contain the right angle. The square of the hypotenuse is equal to the same, namely, four times the area of the triangle, or twice the product of A B and B C together with the square of A B, the difference between A B and B C. While the sum of the sides remains the same, the square of that sum must also remain the same, whatever may be the difference between the two sides themselves, and of their squares as depending upon their respective lengths. If the sides are equal to each other, then the square of the hypotenuse will fall exactly upon the middle of each side of the larger square, that is, its angles will bisect the sum of the sides; and in this state of things the square of the difference will vanish, because the difference itself vanishes. Consequently, the

square of the sum of the sides will be exactly double the square of the hypotenuse, and the square of the hypotenuse exactly double the square of either of the sides, or four times the area of the triangle. There are some conclusions resulting from this which are worthy of attention.

In the first place, we get a general expression for the diameter or diagonal of a square in terms of the side of that square. The diagonal of a square has the same relation to the surface of the square as any diameter of a parallelogram has to the surface of that parallelogram, that is, it divides the surface into two equal parts; but it has another property which does not belong to the diameter of any parallelogram but a square, and therefore it gets a different name, namely the diagonal, which means that which divides into two equal parts each of the angles which it joins. Diameter literally means "through the measure," or dividing the space or surface into two equal parts, while diagonal means "through the angles," or dividing them into equal portions at each extremity. It will be readily understood that, as the sides of the square are equal, the sum of their squares, which is also the square of the diagonal, must be equal to twice the square of one of them. If, therefore, we call the side of the square a, the square of the diagonal will be expressed by $2a^2$, and the diagonal itself by $\sqrt{(2a^2.)}$ Now, $2a^2$ is a product of two factors, the one of which, a^2 is a complete square, and the other, 2, is not. Thus, the product can be resolved into two factors, only one of which, namely the factor 2, is affected by the radical sign; that is, the expression may be reduced to

$$a\sqrt{2}$$

or in words, to find the diagonal of any square of which the side is given, multiply the side by the square root of the number 2; and to find the side of any square of which the diagonal is given, divide the diagonal by the square root of 2. The square root of 2 is an irrational number, incapable of being accurately expressed by any fraction, because 2 is a whole number, not an exact square, and therefore its root cannot be any fractional number whatever. If, however, we find, by the rule formerly given for the extraction of the square root, and extend the operation to five places of decimals, we have the square root of 2 =

1·41421 &c.,

which is near enough for all common purposes, either for finding the diagonal from the side, or the side from the diagonal.

The very same prineiple which leads us to this conclusion, enables us to compare the rectangles of the two segments of a line when it is divided into two equal parts, and also into two unequal ones; or, which is the same thing, it enables us to compare the product of two numbers or quantities with the square of half their sum; and from the mere composition of the figure it will be perceived, that the square of the half sum exceeds the rectangle or product by the square of the half difference; because four times the rectangle or product of the two sides, together with the square of the whole difference, is always equal to the square of the sum, or four times the square of the half sum. If we divide these equals each by four, we obtain quotients having the same ratios to each other as the quantities divided; and as these quantities are equal, the quotients must also be equal; and those quotients are the rectangle or product of the two quantities, together with the square of half their difference in the one case, and the square of half their sum in the other. This principle is very simple; but as it enters into some others which are rather more complicated, it may not be improper to give an illustration of it in general terms.

For this purpose, let $2a$ represent the length of any line whatever, and half the length of that line is of course 1 a, or simply a, and its square is a^2. Let this line be divided into two unequal parts, of which the difference is 2 d, and it is equally evident that the larger segment will be expressed by $a+d$ and the less segment by $a-d$. Multiplying $a+d$ by $a-d$, we obtain a^2-d^2, that is, the square of the half sum exceeds the rectangle or product of the two unequal parts by the square of the half difference.

9. If, in a right-angled triangle, one of the sides containing the right angle is double the other, the hypotenuse is equal to the side which is half the other multiplied by the square root of the number 5.

This also follows from the comparison of the squares of the two sides with that of the hypotenuse, and the perfect equality which, in all cases, subsists between them. If the one side is double the other, its square is four times that of the other, and consequently the sum of their squares, which is equal to the square of the hypotenuse, is five times the square of that other, and the hypotenuse itself is the root of this square. Thus, let a be one of the sides, c the other, and b the hypotenuse, and let $c=2a$; then,—

$$b^2=a^2+c^2\text{; but}$$
$$c^2=4a^2\text{; therefore,}$$
$$b^2=5a^2\text{; and consequently,}$$
$$b=\sqrt{(5a^2.)}$$

Now this last expression for b is resolvable into two factors, one of which, a^2, is a complete square, and the other a number not a square, therefore they may be separated, and we may write

$$b=a\sqrt{5},$$

the meaning of which in words is, that b the hypotenuse, is equal to a, the side which is half the other, multiplied by the square root of the number 5. 5 is a whole number, not a complete square, and therefore its square root can be found only to an approximate value; and this value extended to the fifth place in decimals, is,

$$2 \cdot 23606.$$

9. If to the square of the sum of any two quantities the square of either quantity is added, the sum of these two squares is double that of other two squares, namely, the square of half the quantity where square is not added, and the square of the sum of that half and the quantity which is added.

Thus if a and b are any two quantities whatsoever, then,

$$(a+b)^2 = 2\left(b^2 + \left(\tfrac{1}{2}a+b\right)^2\right);$$

and as a or b may indifferently be the greater quantity, for each of the letters stands for any or all quantities whatsoever, they might be made to change places in their expressions without in the least affecting the truths which the expressions involve.

This proposition might be enunciated in a different manner as follows:

If the one of two quantities is divided into two equal parts, the square of the sum of the quantities, and the square of the undivided quantity are together equal to twice the square of the undivided quantity, and twice the square of the sum of the undivided one and half the divided. This may be shown either generally in respect of all quantities whatsoever, or particularly in respect to geometrical quantities; and as it is a principle of some importance, we shall endeavour to show the truth of it in both ways.

First, generally or algebraically:—Let a and b be the two quantities; and as it is perfectly indifferent which of them we consider as the divided one, let a be divided into two equal parts, and let c represent the half of a; that is, let a equal $2\,c$. Then the truth to be shown is, that

$$(2c+b)^2+b^2=2(c^2+(b+c)^2);$$

and if we can show, by the performance of the requisite multiplications, that those quantities which have the sign of equality between them are equal, that is, reducible to the same expressions, then the truth of the proposition will follow, as a matter of course. Now if we multiply $2c+b$, by $2c+b$, the product is the square, and it is,

$$4c^2+4bc+b^2,$$

add b^2 . . . . . b^2; and

the sum is . . $4c^2+4bc+2b^2$, the half of which

is . . . $2c^2+2bc+b^2$;

and if we can show that twice c^2 together with twice the square of $b+c$ amounts to the same expression, the truth of our proposition will be established. Now the square of $b+c$, or transposing the letters, which does not alter the value as it is still the same sum, $c+b$ is,

$$c^2+2bc,+b^2;$$

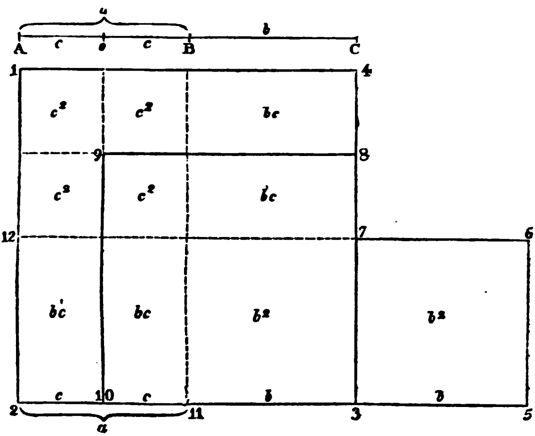
add c^2 . . c^2 and

the sum is . . $2c^2+2bc+b^2$,

which is exactly the same as the former expression; and therefore, the square of the sum, together with that of one of the quantities, is double the square of the other quantity—and the square of the sum of that quantity and half the first one.

Secondly, we shall endeavour to point out the truth of this geometrically, or as it applies to lines; and for this purpose it is usually enunciated in these terms: "If a straight line be bisected, and produced to any point; the square of the whole line thus produced, and the square of the part of it produced, are together double the square of half the line bisected, and of the square of the line made up of the half and the part produced."

In order to show this, we introduce the following diagram :



Let A B and B C be any two straight lines whatsoever, which are placed in the same straight line merely for the sake of convenience; and let the line A B be bisected or divided into two equal parts in the point *o*, so that *c* and *c* are equal to each other and halves of A B, which is marked off by the bracket and named *a* above the same; and B C is named *b* in like manner, in order that the nomenclature may correspond exactly with that previously used in the general reasoning. This being understood, draw any other line, as 2, 5; make *c* and *c*, on each side of 10 equal to halves of A B or of *a*; and make *b* and *b*, one extending to 3, and the other from 3 to 5. 2, 3, is the sum of *a* and *b*, that is, of A B and B C; and 3, 5, is equal to B, that is, to B C. Upon 2, 3, describe the square 1, 2, 3, 4, and it is the square of the sum, that is of $a+b$, and upon 3, 5, beginning where 2, 3, ends, describe the square 3, 5, 6, 7, and it is the square of *b*. Bisect 7, 4, in 8, and through 8 draw 8, 9, parallel to 1, 4, and continue it till it meets 1, 2; also through 10, the bisection of *a*, and 11 the point where *a* and *b* terminate, draw parallels to 1, 2, or to 3, 4, and continue them till they meet 1, 4. The sides of the square 1, 2, 3, 4, are similarly divided into three parts, one part of each being respectively equal to *b*, and each of the remaining two to *c*, that is, to the half of *a*. The lines which cross the square in both directions are parallel to each other, and to the sides of the square; and if the nine parts into which the surface of the square is divided, four and also other four are equal to each other, and they are squares or rectangles accordingly as the sides about any one of them are equal or not equal.

It is not necessary formally to prove the parallelism of the lines, or the equality of the corresponding figures into which they divide the square; for the opposite sides of the square are parallel; and as the lines which intersect the square were made parallel to one of those sides, they must also be parallel to the other. Also, as the angles of the square are right angles, all the angles which the intersecting lines make with each other, or with the sides of the square, must also be right angles; so that the equal-sided portions are necessarily squares, and the unequal sided ones parallelograms.

Let us now examine and see what is the value or surface of both squares, in terms of the lines *c* and *b*. We need not prove that the square 3, 5, 6, 7, is the square of *b*, because it was made the square of *b*. The adjoining square, in which b^2 is marked, is also the square of *b*, for its side 11, 3, was made equal to *b*. Abutting on the dotted sides of this square, there are four rectangles, two upon each side, and they are respectively the rectangles of $b \times c$; for one of the containing sides of each of them was made equal to *b*, and the other equal to *e*. There remain only the four small squares on the reader's left hand, toward the top of the diagram; and as all the sides of each of these is by the construction equal to c, each of them is the square of c.

If we collect the several parts of which the two squares are made up, we have,

First, four times the square of c;
Secondly, four times the rectangle $b\,c$; and
Thirdly, twice the square of b.

If we now examine the square upon 10, 3, which is the sum of c and b, that is, the square 10, 9, 8, 3, we find that it contains, what we formerly showed to be the content of the square of the sum of any or of every two quantities, the sum of their squares, and twice their rectangle or product—in the present case the square of c, the square of b, and twice $b \times c$. If to these we add the square of c, we have the whole equal to

$$2\,c^2 + 2\,b\,c + b^2,$$

which is exactly half the content of the two squares which make up the diagram; and as c is in every instance an expression for half the given line A B, b the expression for the given line B C, and $b + c$ the expression for the line which is made up of the half of A B and the whole of B C, the truth is geometrically apparent.

There are several other truths of a somewhat simple nature, which are apparent from simple inspection of the same diagram; first, (but that was sufficiently explained in a former section,) we have the square of the sum of c and $b = c^2 + 2\,b\,c + b^2$.

Secondly, we have the square of b and the half of a, that is, the square of $b + c =$ the rectangle under $a + b$ and b, together with the square of c. For 2, 3, 7, 12, is the rectangle under $a + b$ and b; and it consists of $2\,b\,c + b^2$; and if to this we add c^2, we have the square of $b + c$, or (which is the same thing) $c + b$, as above stated.

Geometrically, this last truth is usually enunciated: "If a straight line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected is equal to the square of the straight line, which is made up of the half of the part produced."

10. Some very useful problems may be performed by means of the principles which have been explained in the preceding articles; and indeed as squares are the original foundations of our judgment respecting the areas of figures, they are the best means by which figures can be compared in respect of their areas; and as every figure bounded by straight lines is capable of being reduced to a square, this medium of comparison is perfectly general. But we shall proceed to the more useful of the problems.

First, to find a rectangle whose half, fourth, eighth, sixteenth, and so on, proceeding by continual bisection or division by two, shall all be similar to each other, and also to the rectangle itself.

It is obvious, that in order to solve the problem, we must have some datum, and that this datum must be either the length or the breadth of the rectangle; because in every rectangle these two dimensions are quite sufficient for constructing the figure or finding its surface.

Let the length be that which is given, and the problem will be reduced to finding the breadth, in order that half the rectangle may be similar to the whole. Figures are similar when the sides about their equal angles are directly proportional. But the breadth of half the rectangle is evidently half the length; and this half of the length must have the same ratio to the breadth of the rectangle, as that breadth has to the length. Calling the length a, and the breadth x, we have the following proportion:

$$\text{As } a : x = x : \frac{a}{2}.$$

Or multiplying extremes and means, and expressing their products as equals, we have

$$x^2 = \frac{a^2}{2}, \text{ and consequently,}$$

$$x = \sqrt{\frac{a^2}{2}}$$

In words: to find the breadth of the rectangle, divide the square of the length by 2, and the square root of the quotient is the breadth.

But if we examine the expression,

$$x = \sqrt{\frac{a^2}{2}}$$

we can see that the quantity a which is given is quite general, or may be expressed by any number whatsoever, and by the number 1 as correctly as any other; and if we consider a as 1, we shall be enabled to change our expression into a multiplier; for if $a = 1$, then

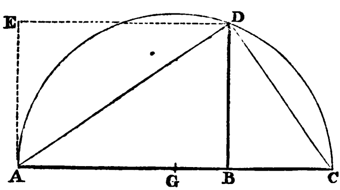
$$x = \sqrt{\frac{a^2}{2}} \text{ becomes } x = \sqrt{\tfrac{1}{2}},$$

for 1 is the square of 1, and therefore the sign of the root may be taken away. Every other quantity when expressed by a number, is expressed by the product of 1 multiplied by the ratio of that number to 1; and therefore we have in the square root of $\frac{1}{2}$, the ratio of the breadth of the rectangle to its length; and if the length is given, we can find the breadth by multiplying the length by the square root of $\frac{1}{2}$. In $\frac{1}{2}$, the numerator 1 is a square number; but the denominator 2 is not, and, therefore, its square root cannot be expressed by any fraction: but it may be approximated decimally with sufficient accuracy for practical purposes. The decimal of $\frac{1}{2}$ is 5·; and if we find the square root of this in the manner explained in the section on roots, we shall find it to be · 7 0 7 1 0 6 &c.; and if we multiply the given length by this number, we shall obtain the breadth. The second figure of the decimal number is 0; and therefore the first figure, namely 7, is as accurate as the first and second 7 0; consequently, if we take $\frac{7}{10}$ as our multiplier, we shall be correct to the nearest 100th part, which is sufficient in many practical cases. Hence we have the popular formula,—

To find the breadth of a rectangle, whose half, fourth, and so on, dividing continually by 2, shall be similar to the whole,—multiply the length by 7, divide by 10, and the quotient is the breadth to the nearest 100th part.

If the breadth is given to find the length, it is merely reversing the operation; that is, multiplying the breadth by 10 and dividing it by 7, and the quotient is the length.

This problem is nothing more than a particular case of the finding of a mean proportional between any quantity, and the half of that quantity; and therefore it may be performed geometrically in the same manner as the general problem which is mentioned at page 445.



Let A B be any straight line; produce it to C, until B C is equal the half of A B; it is required to find a mean proportional between A B and B C.

Bisect A C in G, and from A as a centre, at the distance G A or G C, describe the semicircle A D C. At B draw the perpendicular B D, meeting the circumference in D, and B D is the mean proportional required. Join A D and C D, and the angle A D C is a right angle, being in a semicircle; and the triangles A B D, D B C are similar, and have their sides opposite the equal angles proportional. A B and B D in the one, and B D and B C in the other, are opposite equal angles. Therefore

$$AB : BD = BD : BC;$$

and multiplying extremes and means,

$$AB \times BC = BD^2, \text{ that is,}$$

B D is a mean proportional between A B and B C; and B C was made equal the half of A B, therefore it is a mean proportional between a line and the half of that line.

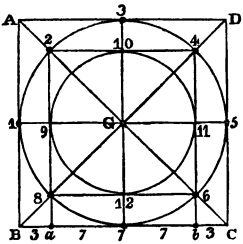
Complete the parallelogram, by drawing the dotted line D E parallel to A B, and A I parallel to B D, and the figure A B D E is a parallelogram, the continual bisection of which will form parallelograms similar to the whole and to each other.

It is not a little remarkable, that this rectangle, which produces similar rectangles upon being bisected, is the most agreeable to the eye of all rectangular figures, not even excepting the square, the perfect regularity of which gives it a tameness, and impresses one with the idea that there is neither up nor down nor right and left about it. But when a rectangle of the proportion above stated is placed with its longer sides perpendicular, as for example when it is a window, a door, or any other rectangular aperture, there is something very satisfactory in its appearance; and if it is contracted with another rectangle which is either longer in proportion to its breadth, or broader in proportion to its length, these are sure to offend the eye even of one who is little accustomed to the study of forms, if they are presented along with one of these proportions. Those that are longer in proportion seem lank and wire-drawn; while those which are shorter in proportion, seem cramped and compressed by their own weight.

It is perhaps not less remarkable, that this proportion of form has been in a great measure, if not altogether, neglected in the arts, especially in modern building, where the rectangle is of such constant occurrence. This single fact shows how ill the technical systems of Mathematics, and more especially of Geometry, are adapted for useful purposes; because if practical men were in the habit of going beyond the merely mechanical use of the problems which they have conned by rote, and availing themselves of the general principles, it is difficult to see how they could by possibility have overlooked one the advantages of which are so great and at the same time so obvious.

There is one particular branch of the arts in which the introduction of this principle would be attended with very pleasing advantages, and that is the manufacture of paper. At present a sheet of paper has of course some sort of shape; and though the shapes of differently named papers vary a good deal, we believe no one can explain why any one of them should have the relative length and breadth which are invariably given to it. The consequence is, that books, printed on different papers, and folded into different numbers of leaves per sheet, are of many different forms, and the greater part of them are offensive to the eye, of which the common demy 12mo is a very convincing instance. Now, if the sheet of paper, of whatever size, were made in the proportion of the rectangle under consideration, or nearly in that of 10 long and 7 broad, then all the foldings of such a sheet into numbers expressing the powers of 2, that is 2 for folio, 4 for quarto, 8 for octavo, and so on till the smallest size necessary for any useful purpose were reached, would be similar forms; and each of them, however large or however small, would be of that form which is most agreeable to the eye. To those who have not reflected upon such subjects, this may seem to be a matter of very minor importance; but in truth it is not; for there is no one object that can come under our notice, in which any thing repulsive to the feelings has more injurious influence than in a book; and any one who thinks as he reads, must remember many instances in which he has felt the matter of the book rendered uninviting and even repulsive by the ungainliness of its shape. It is of course not meant to be alleged that mere shape will convert a dull book into an interesting one; because the pleasure we derive from the contemplation of form is simply a matter of feeling; but there is no doubt that the value of a good book may be greatly deteriorated by an ugly form, just as the most important truth may be made repulsive, if expressed in language which is ungrammatical and vulgar.

There are many other practical applications of the principle involved in this problem; but we shall content ourselves with only one more, which is very useful in the perspective representation of a circle on a plane, oblique, or not every way at right angles to the axis of the eye. In order to draw the perspective representation of the circle with moderate accuracy, it is necessary to determine eight points in the circumference. Four of those points can always be determined by bisecting the square of the diameter by two lines crossing each other; but the four points intermediate between these, namely those in which the circle cuts the diagonals of the inscribed square, or square of the diameter, cannot immediately be determined by the mere application of lines. We shall perhaps make this a little more clear by introducing some diagrams.



Let A B C D be a square, and 1, 5, or 3, 7, the diameter of the larger circle, of which the centre is G, the two diameters 1, 5 and 3 7, and the two diagonals A C and B D, obviously divide the circumference of the circle into eight equal parts in the points 1, 2, 3, 4, 5, 6, 7, 8. If the square A B C D is given, the points 1, 3, 5, 7, are also given, because they are at the middle or bisection of the sides of the square. If the diagonals A C and B D, which are also given if the side of the square is given, are drawn, the other four points, 2, 4, 6, and 8, must be somewhere in those diagonals; but the square itself furnishes no data for pointing out in what place of the diagonals they are situated.

In the case of an actual square, the inscribed circle can readily be drawn, and it will cut the diagonals in the points 2, 4, 6, and 8; but when we wish to make a perspective representation of a circle in an oblique plane, the circle itself is the thing sought, and therefore, instead of using the circle to find the points, we must use the points to find the circle.

Join 2, 4, 6, and 8, by four lines; and the figure which they form, as its sides stand upon four equal arcs, each a quadrant or quarter of the circle, the figure 2, 4, 6, 8, is a square, having its sides parallel to those of the square without the circle. The diameters 1, 5, and 3, 7, divide both the circumscribing and the inscribed square, that is, the square touching the circle within into four equal squares; and the diagonals of the squares divide each of those four squares into two equal triangles; so that the eight parts which have each an angle at the centre G, are all equal as belonging to the one square or to the other; that is to say, the triangle A 1 G is one eighth of the larger or circumscribing square, and the triangle 2 9 G is one eighth of the smaller or inscribed square.

We may therefore compare the two squares, both in their areas and in the lengths of their sides, in terms of these two triangles, because like parts are proportionals.

Now the triangles A 1 G, and 2, 9 G, are similar, having their angles at 1 and 9 right angles, and the sides in each containing the right angle are equal; and therefore the square of the hypotenuse in each is equal to twice the square of a side about the right angle, that is,

$$\text{A G}^2 = 2 \times 1\,\text{G}^2, \text{ and } 2\,\text{G}^2 = 2 \times 9\,\text{G}^2; \text{ or}$$

$$1\,\text{G}^2 = \tfrac{1}{2}\,\text{A G}^2, \text{ and } 9\,\text{G}^2 = \tfrac{1}{2}\,2\,\text{G}^2;$$

but 1 G and 2 G are equal, being radii of the same circle, and therefore $1\,\text{G}^2 = 2\,\text{G}^2$, consequently $9\,\text{G}^2 = \frac{1}{2}\,1\,\text{G}^2$, and as $1\,\text{G} = \frac{1}{2}$ of A G^2, 9 G is to 1 G, as the square root of $\frac{1}{2}$ to the square root of 1. But we have shown that the approximate value of the square root of $\frac{1}{2}$ is ·7, that is, $\frac{7}{10}$, 9 G contains 7 of those parts of which 1 G contains 10, and consequently 1,9 contains the remaining 3 of the 10 parts.

Therefore on any side of the larger square, which is the only square given in the case where this problem applies in perspective; as, for example, in the side B C, divide each half, that is, 7 B, and 7 C, into 10 equal parts. Set off 7 of those parts from the point 7, to the two points *a* and *b*, and there will remain 3 parts from *a* to B, and other 3 from *b* to C. Through *a* and *b* draw parallels to A B or C D; and those parallels will cut the diagonals, and determine by their intersection the points 2, 4, 6, and 8, which are all that are required for the perspective delineation of a circle on an oblique plane.

From the mere inspection of the above figure, and by only drawing in it a circle touching the smaller square in the points 9, 10, 11, 12, we can perceive some of the general relations of figures without formal investigation.

The first which occurs is one which we have already mentioned, namely, that circles are to each other in the ratio of the squares of their diameters; and it immediately follows from this, that a circle described on the hypotenuse of a right-angled triangle as diameter, is equal to the sum of the two circles described on the sides. Again, this correspondence between circles and the squares of their diameters, is not confined to squares; but applies to all similar figures described in, or inscribed about circles; namely, the corresponding sides of those similar figures are to each other in the ratio of the squares of the diameters of the circles about which they are described, or in which they are inscribed, it being understood that the two which are compared must either be both inscribed, or both described.

It follows from this still farther, that if similar figures, of any form whatsoever, are described on the three sides of a right-angled triangle, the figure upon the hypotenuse is, in all cases, exactly equal to the sum of the figures upon the sides about or containing the right angle.

Hence, still farther, it is easy to see that, if any similar figures whatever which are described upon three lines, the square of one of which is equal to the squares of both the other two, the figure, of whatever form, upon the first, is equal to the two similar figures upon the other two; those figures must be to each other in the ratio of the squares of their corresponding sides; and thus if we once get possession of the whole particulars of one rectilineal figure, we can get the particulars of every similar figure, by a simple application of the rule of three, if we merely know the ratio between one part of the figure which we are seeking to know, and the corresponding part of the similar figure of which we already know the particulars.

This comparison of all rectilineal figures through the medium of the square, is one of the best and most extensive principles in mathematics; inasmuch as it is the one which more especially brings the practical applications of geometrical truth within the range of common arithmetic; and further than this, we have only to show how all rectilineal figures may be reduced to similarity, or to rectangles or squares, so that they may be compared with the same ease and the same accuracy as lines and angles, and figures, which are similar according to the conditions of those which are given. It is this possibility of comparing all straight-lined figures with each other, through the medium of the square, which renders the whole of them—how different soever they may be in shape—quantities of the same kind, and therefore always proportionals, whether we happen to know the proportions of them or not. We must, however, in the mean time, advert to our other problem:—Second, to divide any quantity or line into two such parts, as that the square one part shall be equal to the product or rectangle of the whole and the other part.

We shall consider this problem as we did the other, both generally and algebraically, and particularly as it applies to lines in geometry. In the latter branch of the science, the operation here to be performed is sometimes called dividing a straight line in extreme and mean ratio, which means, that if a proportion is instituted, the whole line shall be either the first or fourth term, it is of no consequence which, and one of the parts the other extreme, that is, either the fourth or the first term, and that the other part of the line shall be both second term and third term, and if they have this property as proportionals, it is evident that the square of the part which stands for the second term, and the third must, upon the common property of proportionals, be equal to the rectangle or product of the whole and the other part.

The truth of this is very plain, as well as very obvious; but when we come to attempt the practical solution of the problem, we find it beyond the reach of the principles of common arithmetic. Only one quantity is given us, and we are required to find two quantities whose sum shall be equal to the given one, and which shall be such, that the product of the whole by one of them shall be equal to the square of the other.

But we are not furnished with either the factors of the product, or the side of the square which is equal to it; nor have we the value of the product, or square itself; so that we are obliged to have recourse wholly to ratios; these ratios are, however, quite sufficient for our purpose.

Let a be any quantity whatever, and let it be required to divide a into two such parts, as that the product of a by one of them shall be equal to the square of other one.

We must at first content ourselves with expressions for those two parts of a, so that from these we may get at their relations to a, and to each other; so let one of the parts be called x, and the other, which is the remainder of a after x is taken away, is of course explained by $a-x$. x, and $a-x$ are our substitutions for the two parts; and as we do not at the present know the relation of either part to a as a measure, it is impossible for us to tell whether x is greater or less than $a-x$. That does not affect the truth of the result, however; because there can be only one value of x answering to every value of a.

The conditions are,

$$x^2 = a \times (a - x); \text{ and}$$

performing the multiplication,

$$x^2 = a^2 - a x.$$

From this we can discover that the square of x is equal to what remains after the product of a and x is taken away from the square of a; and this is something, because it shows us that the square of a is, by a parallel drawn through the point of division which we are seeking, divided into two rectangles or products, one of which is $a x$, and the other equal x^2. This, however, does not give us x itself, or any means of arriving at it; but it enables us to state our expression differently; so as to have only a^2 on the one side of the sign $=$; for since a^2 is the sum of x^2 and $a x$, we have

$$x^2 + a x = a^2.$$

In this case we could readily obtain the square root of a^2; but we cannot directly find that of $x^2 + a x$, because it is not a complete square. It contains two quantities, x and a, and in order to be the square of those two quantities, it should consist of three terms; for according to the composition of the square of a binomial, or quantity consisting of two terms, as explained in a former section, such a square is always

$$a^2 \pm 2 a b + b^2.$$

We use the double sign $\pm$, in order that the expression may be perfectly general, and represent the square of $a - b$, as well as that of $a + b$; and with this understanding let us compare the general formula with our data in this particular case:

General formula . . . . . . $a^2 + 2 a b + b^2$
Particular case . . . . $x^2 + a x + *$.

Here we find x^2 corresponding to a^2, which will do, as they are both complete squares. Then, in the second term we find $a x$ corresponding to $2 a b$, which may also do, if we consider a as representing $2 b$, and x as representing the other factor, a.

In this case, we shall get an equivalent expression to b in the general formula if we divide a by 2, and our second term will then become $2\frac{ax}{2}$. The third term is, however, altogether wanting in our expression, but we can see what it ought to be from the general formula. In that it is b^2, and our equivalent for b is $\frac{a}{2}$, consequently, our equivalent for b^2 is the square of $\frac{a}{2}$, that is, $\frac{a^2}{4}$. Therefore the terms preceding the sign = are reduced to

$$x^2+ax+\frac{a^2}{4},$$

which is a complete square; but it is greater than the original expression x^2+ax by the third term $\frac{a^2}{4}$. To preserve the equality, we must add this to the other side of the sign = in our expression, and we shall have

$$x^2+ax+\frac{a^2}{4}=a^2+\frac{a^2}{4}.$$

But the two terms after the sign = may be added together; for a^2 is $\frac{4a^2}{4}$, and if to this we add $\frac{a^2}{4}$, the sum is $\frac{5a^2}{4}$. Therefore,

$$x^2+ax+\frac{a^2}{4}=\frac{5a^2}{4}.$$

The three terms preceding the sign = are a complete square, namely, the square of $x+\frac{a}{2}$, for they were made such; and as the squares of equal quantities are equal, and the square roots of equal squares also equal, we may take the square roots of those quantities as far as possible. That of the terms before the sign $=$, is $x+\frac{a}{2}$; but the quantities after the sign do not make a complete square, and therefore we must, in the mean time, merely indicate the root by the sign $\sqrt{}$, thus,

$$x=\frac{a}{2}=\sqrt{\frac{5a^2}{4}}.$$

If we take $\frac{a}{2}$ from each of those equals, we get equal remainders, and we leave nothing but x before the sign $=$, so our expression becomes,

$$x=\sqrt{\frac{5a^2}{4}}-\frac{a}{2},$$

which gives us the value of x in terms of a, or rather of the half of a. Let us now see how we can simplify this value of x, namely,

$$\sqrt{\frac{5a^2}{4}}-\frac{a^2}{2}.$$

The denominator of 4 is a square; and so is the factor a^2 of the numerator; so that, as formerly explained, we can take their roots, and remove them from under the radical sign, which will reduce our expression to,

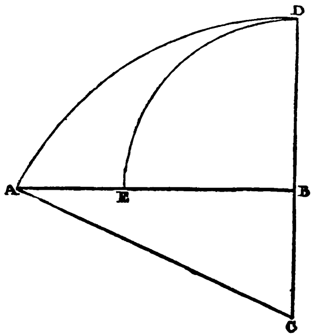
$$x=\sqrt{5}\times\frac{a}{2}-\frac{a}{2},$$

that is to say, in words, the greater segment of the line, and that whose square is equal to the product of the whole line by the other one, is obtained by multiplying half the line by the square root of 5, and subtracting half the line from the product, or, more simply, by multiplying half the line by the square root of 5, after 1 is subtracted from it, that is by $\sqrt{5}-1$; and as this is an operation easily performed, we can at once find the parts of the quantity, or the segments of the line, if expressed in numbers.

The simplest numerical expression which we can have for any quantity is 1; and if we use this in place of $\frac{a}{2}$, our expression becomes

$$\sqrt{5}-1.$$

We are now in a condition for showing how the same result may be arrived at geometrically.



Let A B be any line, it is required to divide it into two parts, so that the square of one part shall be equal to the rectangle of the whole and the other part. At the point B, the extremity of A B, draw B C at right angles, make B C equal to the half of A B, and produce it indefinitely toward D on the other side of B; also join A B, and A B C is a right-angled triangle, having the sides B C and A B in the ratio of 1 and 2, therefore C A is in the ratio of the square root of 5. On C as a centre, with the distance C A, describe an arc meeting C D in D; and B D is equal to that part of the given line A B whose square is equal to the rectangle of the whole and the other part. On B as a centre, and with the radius B D describe an arc cutting A B in E; and A B is divided in E, so that the square of B E is equal to the rectangle under the whole line A B, and the other part A E.

Because C is the centre of the arc, C D is equal to C A, and B C is equal to the half of A B; therefore C A is to B C, as the square root of 5 to 1; but B D is the remainder of C D after B C is taken away, therefore B D is equal to the square root of 5 *minus* 1, which is the part required; and again, because B is the centre of the arc D E, B E is equal to B D; and consequently A B is divided in the ratio which was required.

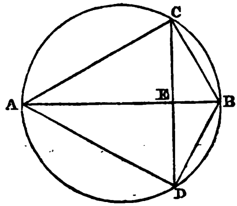
LINES AND CIRCLES.

In order to bring the circle properly within the province of elementary geometry, it is necessary to have the means of determining the length of the circumference and diameter in terms of each other, or in other words to find their ratio. We formerly stated the approximate value of this ratio; and we are not now to enter farther into the investigation of the circle, than merely to state one or two properties of some lines in and about a circle, which have their relations determined by that figure.

1. If two straight lines cross or cut each other in a circle, the rectangle or product of the two segments of the one is equal to that of the two segments of the other, without any regard whatever to the lengths of the lines, or their situation in the circle, provided that they cross each other, and the extremities of both are on the circumference.

If both lines pass through the centre, the truth asserted is self-evident; because each of the segments of both is a radius of the same circle, and consequently they and their products or rectangles two and two, are equal.

There are other three cases, one line passing through the centre, crossing the other at right angles, and consequently bisecting it; and the proof of this is also nearly self-evident.



Let A B, passing through the centre, cut C D, which does not pass through the centre, at right angles, in the point E, then the rectangle A E × E B is equal to the rectangle C E × E D, that is, as C D is bisected the square of C E or E D.

Join A C, C B, B D, D A; and the triangles A C B, A D B, are equal to each other, and right-angled at C and D, because each of them is a semicircle. C E and D E, the segments of C D, are the perpendiculars drawn from the right angles of those triangles to the opposite side; therefore each of them is a mean proportionable between A E and E B, the segments of A B their common base; consequently,

A E : E C = E D : E B;

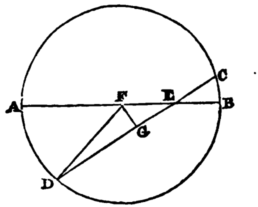
and, multiplying extremes and means,

A E × E B = C E × E D, that is,

the rectangle of the segments of A B is equal to that of those of C D.

Next, let A B, which passes through the centre, cut C D, not at right angles, and consequently not bisect it, the rectangle of the segments of the one is still equal that of the segments of the other.

Let A B pass through the centre, and cut C D in the point E, but not at right angles, and consequently not into equal parts.



Find the centre F by bisecting A B, and from F draw A G at right angles to C D, and F D. C D is equally divided in G, and unequally in E, and A B is equally divided in F and unequall in E; therefore,

$$DG^2 = DE \times EC + GE^2.$$

Add the square of F G to both, and

$$DG^2 + FG^2 = DE \times EC + GE^2 + GF^2.$$

But because F is the centre of the circle, D F is equal A F or F B; and because of the right angle at G, $DF^2 = DG^2 + GF^2$; and because D F is equal to A F, and A F to F B, the square of $DG + GF = AE \times EB + FE^2$, but $FE^2 = FG^2 + GE^2$; therefore $AE \times EB + FG^2 + GE^2 = DG^2 + FG^2$.

But it has been already shown, that

$$DG^2 + FG^2 = DE \times EC + GF^2 + GF^2 : .$$

Therefore—

$$AE \times EB + FG^2 + EG^2 = DE \times EC + GF^2 + GF^2.$$

Take away the parts $FG^2 + EG^2$, which are common t both; and the remaining rectangles are equal, that is,

$$AE \times EB = DE \times EC,$$

and they are the segments of the lines.

If neither of the lines passes through the centre of the circle, a line can be drawn through the point of section and the centre, meeting the circumference both ways; and as it can be shown (as in the last case) that the rectangle of the two segments of each of the lines, is equal to the rectangle of the segments of this one which passes through their point of section and the centre, it follows that their rectangles must be equal to each other.

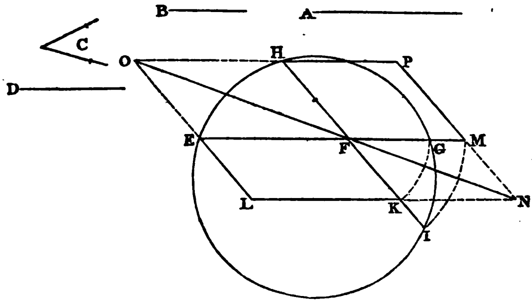
2. As the rectangles of the segments of any two lines which intersect each other in a circle are equal, it follows conversely that lines which intersect each other so as to make the rectangles of the segments of each line equal, are so situated that a circle can always be described so as to touch all their four extremities; and the centre of this circle can in every case be found by bisecting each of the lines at right angles; for, as both the bisecting lines must pass through the centre, that centre must be the point where the perpendiculars intersect each other.

But it also follows, that when the rectangle formed by two lines is equal to that formed by other two, and the contained angle is equal in both, the sides about the equal angles must be reciprocally proportional, that is, in looking back to either of the last figures,

A E : D E = E C : E B.

Hence we have a general method of transforming any rectangle or parallelogram whatsoever into another of the same surface, but differing in the lengths of the sides and their ratios to each other; for we have only to place the sides of the given rectangle in a straight line, to draw from the point where they meet a line the length of one of the proposed sides of the equal but not similar figure; and by describing a circle through the three points, and producing the line which was made equal to one of the sides of the proposed figure until it meets the circumference, in order to get the remaining side of the rectangle sought. It may be worth while to illustrate this by a diagram.

Let A and B represent two sides of a parallelogram, and C one of the angles which it contains, and let D be one side of another parallelogram which is to have the same area and the same angles, it is required to find the remaining side of this parallelogram.



Make the line E F equal to A, and continue it to G till F G is equal to B. At the point F draw the line F H, equal to D, and making the angle E F H equal to C. Through the points E H and G describe a circle, and produce H F till it meet the circumference of this circle in the point I, and F I is the side required.

To complete the parallelograms, transfer F G to F I by the arc G K, and transfer F I to F M in F G produced, and the sides I F, F K, and H F, F M, are the sides of the equal parallelograms, only they contain the supplements of the proposed angle C; complete the parallelograms, by drawing parallels meeting in L and in P, and I F K L and H F M P are the equal parallelograms, having I F : H F = F M : F K, which were required; continue the parallels till they meet in O N as marked by the dotted lines, and the whole figure O L N P is a parallelogram; and if O N are joined, O N is a diameter of that parallelogram, and it passes through F, the point in the circle where E G and H I intersect each other.

The sides of the parallelograms E F K L and H F M P, which are about their equal angles at F, are reciprocally proportional, and for this reason they are directly proportional in the reverse parallelograms I F H O and M F K N, of which they are also sides, and contain the supplements of the former angles, namely E F H and M F K. that is,

$$IF : FH = MF : FK;$$

wherefore, those two parallelograms are similar to each other; but they are also similar to the whole parallelogram O L N P; for its sides O L and O P are respectively the sums of H F + F K, and E F + F M, and these sums are proportional to E F and F H, or M F and F K taken singly, proportionals being proportional by addition of their corresponding terms. Therefore, the line O N bisects the whole parallelogram O L N P, and the two parts of it, O E F H and M F K N.

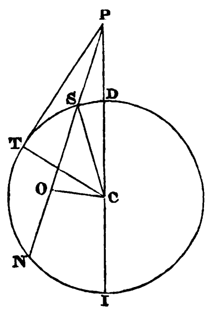
The parallelograms which have the sum of their diameters equal to the diameter of a parallelogram containing them, and their sides parallel to its sides, are called parallelograms about the diameter of a parallelogram; and the remaining parts are called complements to these. In the case of two parallelograms about the diameter, as in the present figure, there are two complements; but whatever number there are, the corresponding complements are always equal to each other, and the parallelograms about the diameter are similar to the whole and to each other.

By means of these correspondencies, it is easy to change a parallelogram to another having its sides in any proportion, by making it a complement to the given proportional; and it is equally easy to make any change in the angles without changing the area; for we have only to put the figures upon equal bases and between parallels equally distant from each other. The same facility of change applies to triangles; for a triangle is always half the parallelogram on an equal base, and between parallels equally distant from each other; and thus we can apply triangle after triangle to any given line, so as to form the whole into one parallelogram, having that line for one of its sides, and one of its angles equal to any angle that may be proposed. After we have obtained the parallelogram, we have only to change its angles or its sides by the methods already explained, in order to reduce it to a parallelogram having its sides in any proportion, or into a square.

As we can perform all these operations directly we can also perform them inversely, and resolve a square into any rectangle, parallelogram, triangle, or number of triangles, that we please, or into any other straight line figure whatever; but these transformations are so seldom needed in practice, that it is unnecessary to go into any details of them; for any one who studies with attention what has been stated, can find no difficulty in performing for himself any or all of these transformations; and they involve the whole principles necessary for the comparison of every form of rectilineal figure, which is the utmost extent to which we purpose to carry the elements of Geometry, in the present volume. There is still however one determination of the relations of lines by means of the circle to which it may not be improper to advert; and that we shall now briefly consider.

3. If from any point, without the circumference of a circle, two straight lines be drawn, one of which cuts the circle and the other touches it, the rectangle contained by the whole line which cuts, and the part of that line without the circle, is equal to the square of the line which touches.

The line which touches the circle is always of the same length, if the point from which it is drawn is fixed, that is, if the position of it is given; for it can touch the circle only in one point, which is fixed also because the position of the circle is given; and therefore the only variation which can be is in the position of the line which cuts the circle. It may pass through the centre of the circle, or it may not; but in what manner soever it cuts the circle, while it is drawn from the same point without the rectangle under the whole of it, and the part without the circle remains constantly the same, or equal to the square of the line which it touches; and this line itself remains unchanged, while the point in the circle remains the same. There is, therefore, the same relation between the whole of a line which cuts a circle from a point without, and the part of it which is without, that there is between the segments of a line which cuts another line in the same point within a circle, that is, the rectangle between them is of the same value however their lengths may vary, and constantly the one increases in proportion as the other diminishes; and if we take two lines drawn from the same point without, and both cutting a circle, the whole and the part without of the one must be reciprocally proportional to the whole and the part without of the other. If we show the truth of this in the case of one line which passes through the centre, and of another line which does not, we shall have proved all that is necessary.



Let there be any circle of which C is the centre, and any point P without the circle. Let there be drawn from P, a line P T touching the circle in the point T, and two lines P L, P N, both cutting the circle, but P I passing through the centre and

P N not, the rectangle under the whole P I, and the part without the circle P D, and the rectangle under the whole P N, and the parts without P S, are each equal to the square of P T, and, consequently, they are equal to each other.

First, let us consider P I. Join T C, and because P T touches at T, and T C from the point of contact passes through the centre, P T C is a right angle, and

$$PC^2 = PT^2 + TC^2.$$

But I D is bisected in C and produced to P, therefore

$$PI \times PD + DC^2 = PC^2.$$

But $DC^2 = TC^2$, for both lines are radii, therefore

$$PI \times PD + TC^2 = PT^2 + TC^2.$$

Take away TC^2, which is common to both, and

$$PI \times PD = PT^2.$$

Second, $P \times PS$ is also equal to PT^2. Join C S, and from C draw C O at right angles to P N, and the part S N is bisected in O, therefore,

$$PN \times PS + SO^2 = PO^2.$$

To each of these add OC^2, and

$$PN \times PS + SO^2 + OC^2 = PO^2 + OC^2.$$

But $PC^2 = PO^2 + OC^2$, and $CS^2 = S^2 + OC^2$; therefore, $PN \times PS + OC^2 = PC^2$; and $PC^2 = TC^2 + TP^2$, and also to $SC^2 + PT^2$, take away the equals and $PN \times PS = PT^2$, that is, the rectangle under the whole line and the part without the circle, is equal to the square of the touching line, wheth the line which cuts does or does not pass through the centre.

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